Kees, C. Vuik

Deflation acceleration of the preconditioned Conjugate Gradient method

Delft University of Technology
Department of Applied Mathematical Analysis
Mekelweg 4
2628 CD Delft
The Netherlands
c.vuik@math.tudelft.nl
Reinhard, R. Nabben
Jarno, J. Verkaik

It is well known that the convergence rate of the Conjugate Gradient method is bounded as a function of the condition number of the system matrix to which it is applied. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite. We assume that the vector $b \in \mathbb{R}^n$ represents a discrete function on a grid Ω and that we are searching for the vector $x \in \mathbb{R}^n$ on Ω which solves the linear system

$$Ax = b.$$

Such systems are encountered, for example, when a finite volume/difference/element method is used to discretize an elliptic partial differential equation defined on the continuous analog of Ω.

If the condition number of A is large it is advisable to solve, instead, a preconditioned system $M^{-1}Ax = M^{-1}b$, where the symmetric positive definite preconditioner M is chosen such that $M^{-1}A$ has a more clustered spectrum or a smaller condition number than that of A. Furthermore, M must be cheap to solve relative to the improvement it provides in convergence rate. With respect to the known preconditioners at least two problems remain:

- if there are large jumps in the coefficients of the discretized PDE the convergence of PCG becomes very slow,
- if a block preconditioner is used in a domain decomposition algorithm the condition number of the preconditioned matrix deteriorates if the number of blocks increases.

Both problems can be solved by a deflation technique or a suitable (additive) coarse grid correction. In this paper we describe and compare both methods. We also compare deflation with the multiplicative coarse grid correction and the balanced Neumann-Neumann method.

It appears that for the convergence speed of all methods, the choice of the projection vectors is very important. Some well known choices are: (approximate) eigenvectors and subdomain deflation vectors. In the second choice we assume that the computational domain is subdivided into m subdomains. The m deflation vectors are equal to one in one subdomain and zero in the other subdomains. Recently a generalization of this approach has been investigated. In this generalization, deflation vectors on each subdomain are added, which are linear in a coordinate direction. We observe that the number of iterations and the CPU time decreases considerably, if these deflation vectors are used. The reasons are: the norm of the initial residual is much smaller and the rate of convergence is higher.

Finally, problems can occur if the matrix A is symmetric positive semidefinite (this happens if only Neumann boundary conditions are used). In this case subdomain deflation vectors lead to a singular coarse grid matrix. In this paper we compare two approaches to solve this problem.