On the Convergence of Shifted Laplace Preconditioner Combined with Multigrid Deflation for the Helmholtz equation

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The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x,y) - k^2(x,y)\mathbf{u}(x,y) = \mathbf{g}(x,y) \text{ in } \Omega$$

$\mathbf{u}(x,y)$ is the pressure field,
$\mathbf{k}(x,y)$ is the wave number,
$\mathbf{g}(x,y)$ is the point source function and
Ω is the domain. Absorbing boundary conditions are used on Γ.

$$\frac{\partial \mathbf{u}}{\partial n} - \iota \mathbf{u} = 0$$

n is the unit normal vector pointing outwards on the boundary.

Problem description

- Second order Finite Difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4-k^2h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $Au = g$: properties
  Sparse & complex valued
  Symmetric & Indefinite for large k
- Is traditionally solved by a Krylov subspace method, which exploits the sparsity.

Preconditioning

- ILU and variants
- From Laplace to complex Shifted Laplace Preconditioner (2005)
- Shifted Laplace Preconditioner (SLP)

$$M := -\Delta\mathbf{u} - (\beta_1 - \iota\beta_2)k^2\mathbf{u}$$

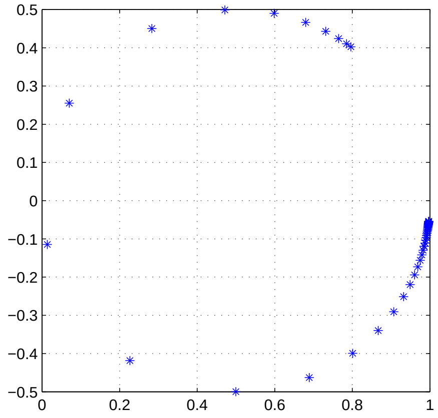
- Results shows: $(\beta_1, \beta_2) = (1, 0.5)$ is the shift of choice
- What is the effect of SLP?

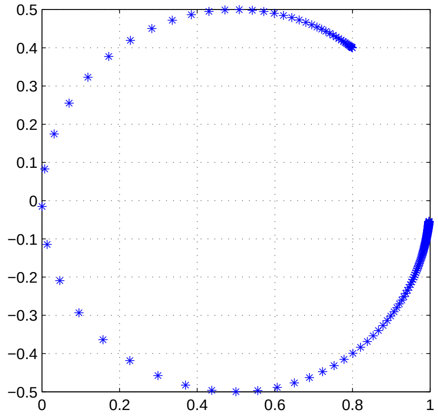
Shifted Laplace Preconditioner

- Introduces damping, Multi-grid approximation is possible
- The modulus of all eigenvalues of the preconditioned operator is bounded by 1
- Small eigenvalues move to zero, as k increases.

Spectrum of $M^{-1}(1, 0.5)A$ for

$k = 30$ and $k = 120$





Some Results at a Glance

Number of GMRES iterations. Shifts in the preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**10**	17	28	44	70	14
$n = 64$	10	**17**	28	36	45	163
$n = 96$	10	17	**27**	35	43	97
$n = 128$	10	17	27	**35**	43	85
$n = 160$	10	17	27	35	**43**	82
$n = 320$	10	17	27	35	42	**80**

Number of iterations depends linearly on k.

Deflation improves the convergence

Number of GMRES iterations. Shifts in the preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 64$	4/10	**6/17**	8/28	12/36	18/45	173/163
$n = 96$	3/10	5/17	**7/27**	9/35	12/43	36/97
$n = 128$	3/10	4/17	6/27	**7/35**	9/43	36/85
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Erlangga and Nabben, 2008, seems to be independent of k.

with / without deflation.

Erlangga and Nabben algorithm

Setting up:

For $k = 1$, set $A^{(1)} = A$, $M^{(1)} = M$, construct $Z^{(1,2)}$, $\lambda_{max}^{(k)} = 1, \ \forall \ k$.

From above, $\hat{A}^{(1)} = A^{(1)} {M^{(1)}}^{-1}$ and $P^{(1)}_{\lambda_{max}} = I - \hat{Q}^{(1)} \hat{A}^{(1)} + \hat{Q}^{(1)}$ with $\hat{Q}^{(1)} = Z^{(1,2)} {\hat{A}^{(2)}}^{-1} {Z^{(1,2)}}^T$

For $k = 2, \ldots, m$, construct $Z^{(k-1,k)}$ and compute

$$A^{(k)} = {Z^{(k-1,k)}}^T A^{(k-1)} Z^{(k-1,k)}, \ M^{(k)} = {Z^{(k-1,k)}}^T M^{(k-1)} Z^{(k-1,k)}$$

and

$$P^{(k)}_{\lambda_{max}} = I - Z^{(k-1,k)} {\hat{A}^{(k)}}^{-1} {Z^{(k-1,k)}}^T \left(\hat{A}^{(k)} - I \right) \text{ with } \hat{A}^{(k)} = A^{(k)} {M^{(k)}}^{-1}$$

Inside Iterations

Solve: $A^{(2)}{M^{(2)}}^{-1}v_R^{(2)} = (v_R)^{(2)}$ with Krylov

$v_A^{(2)} = A^{(2)}v^{(2)}$;

$s^{(2)} = {M^{(2)}}^{-1}v_A^{(2)}$;

$t^{(2)} = s^{(2)} - \lambda_{max}^{(2)}v^{(2)}$;

Restriction: $(v_R)^{(3)} = {Z^{(2,3)}}^T t^{(2)}$

If $k = m$

$v_R^{(m)} = {A^{(m)}}^{-1}(v'_R)^{(m)}$

else

Solve: $A^{(3)}{M^{(3)}}^{-1}v_R^{(3)} = (v_R)^{(3)}$ with Krylov

$\ldots$

Interpolation: $v_I^{(2)} = Z^{(2,3)}v_R^{(3)}$

$q^{(2)} = v^{(2)} - v_I^{(2)}$

$w^{(2)} = {M^{(2)}}^{-1}q^{(2)}$

$p^{(2)} = A^{(2)}w^{(2)}$

Deflation: or two-grid method

For any deflation subspace matrix

$$Z \in R^{n \times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rankZ = r$$

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \quad \text{and} \quad E = Z^T AZ$$

Solve $PAu = Pg$ preconditioned by M^{-1} or $M^{-1}PA = M^{-1}Pg$

For e.g. say,

$$\textbf{spec}(A) = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n\}$$

and if Z is the matrix with columns the r eigenvectors then

$$\textbf{spec}(PA) = \{0, ..., 0, \lambda_{r+1}, ...\lambda_n\}$$

Deflation

We use multi-grid inter-grid transfer operator (Prolongation) as deflation matrix.

Setting $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

$$P = I - AQ, \quad \text{with} \quad Q = I_h^{2h} E^{-1} I_{2h}^h \ \text{and}\ E = I_{2h}^h A I_h^{2h}$$

where

P can be interpreted as a coarse grid correction and

Q as the coarse grid operator

Fourier Analysis

Dirichlet boundary conditions for analysis.

With above deflation,

$$\textbf{spec}(PM^{-1}A) = f(\beta_1, \beta_2, k, h)$$

is a complex valued function.

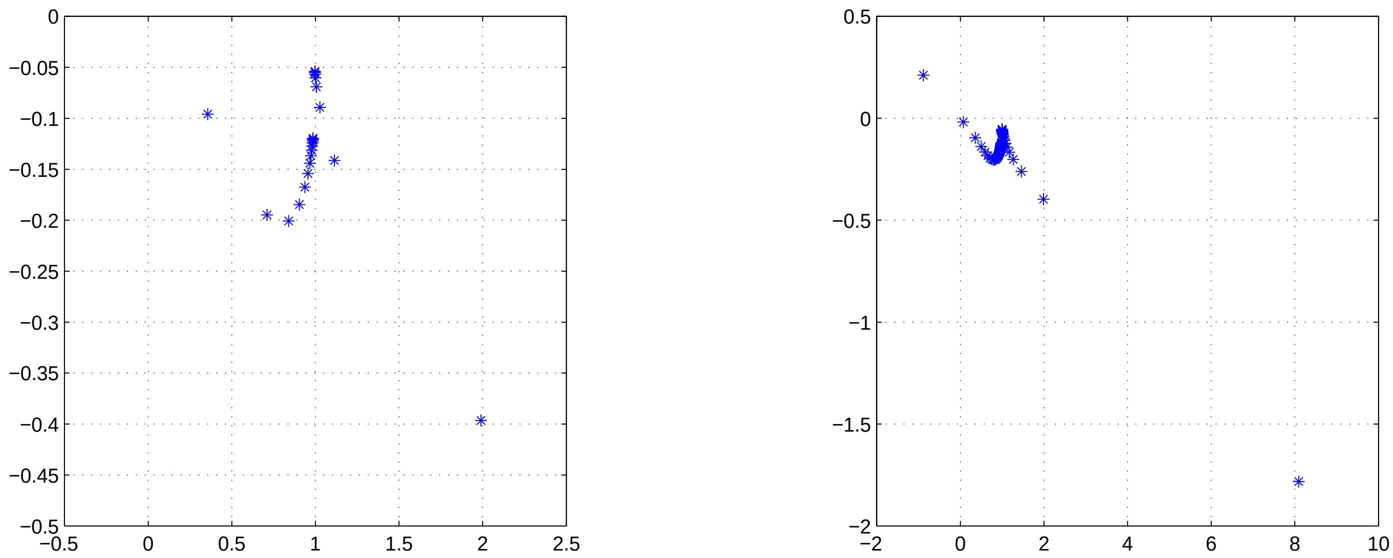
Setting $kh = 0.625$,

- Spectrum of $PM^{-1}A$ with shifts $(1, 0.5)$ is clustered around 1 with a few outliers.
- Spectrum remains almost the same, when the imaginary shift is varied from 0.5 to 1.

Fourier Analysis

Analysis shows spectrum clustered around 1 with few outliers.

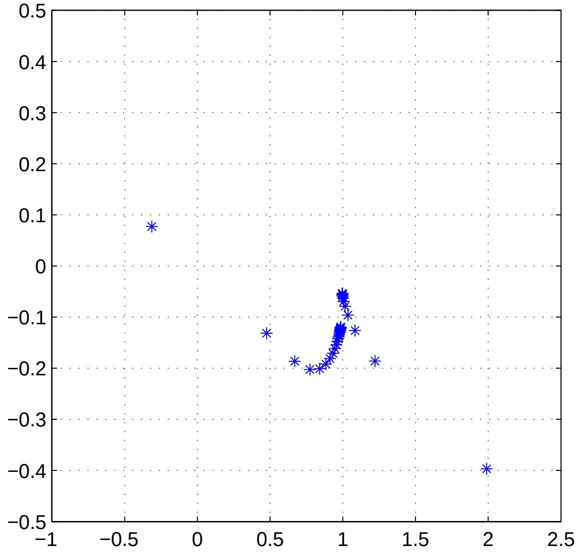
$k = 30$ $\qquad$ $k = 120$



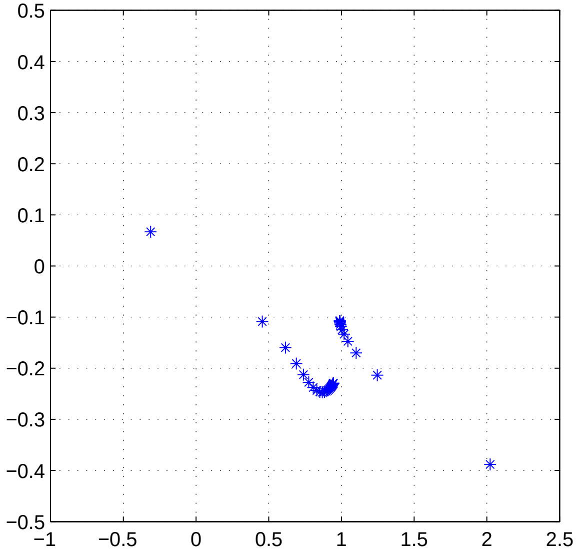
Fourier Analysis

Analysis shows that an increase in the imaginary shift does not change the spectrum.

$(\beta_1, \beta_2) = (1, 0.5)$



$(\beta_1, \beta_2) = (1, 1)$



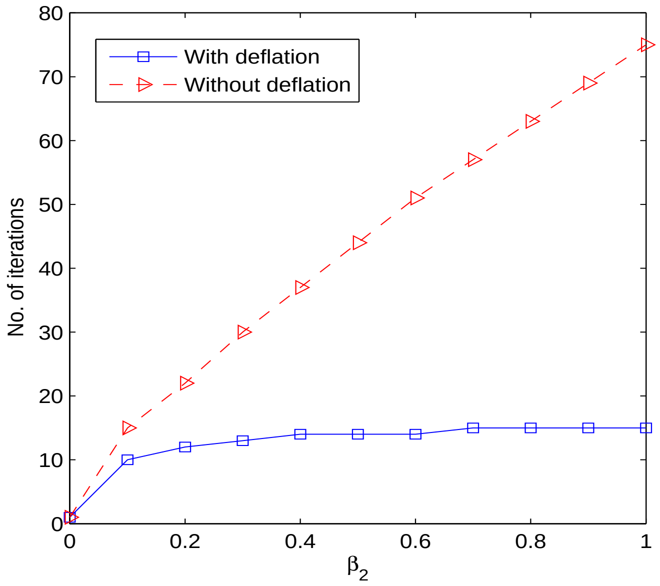
Numerical results

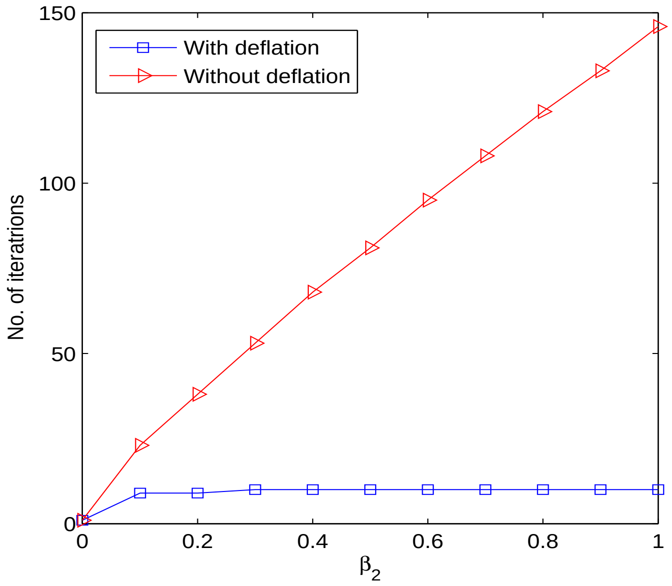
Sommerfeld boundary conditions are used for test problem.

What is the effect of an increase in the imaginary shift in SLP?

Constant wavenumber problem

Wedge problem





Numerical results

Number of GMRES iterations with/without deflation. Shifts in the preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
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$n = 96$	3/10	5/17	**7/27**	9/35	12/43	36/97
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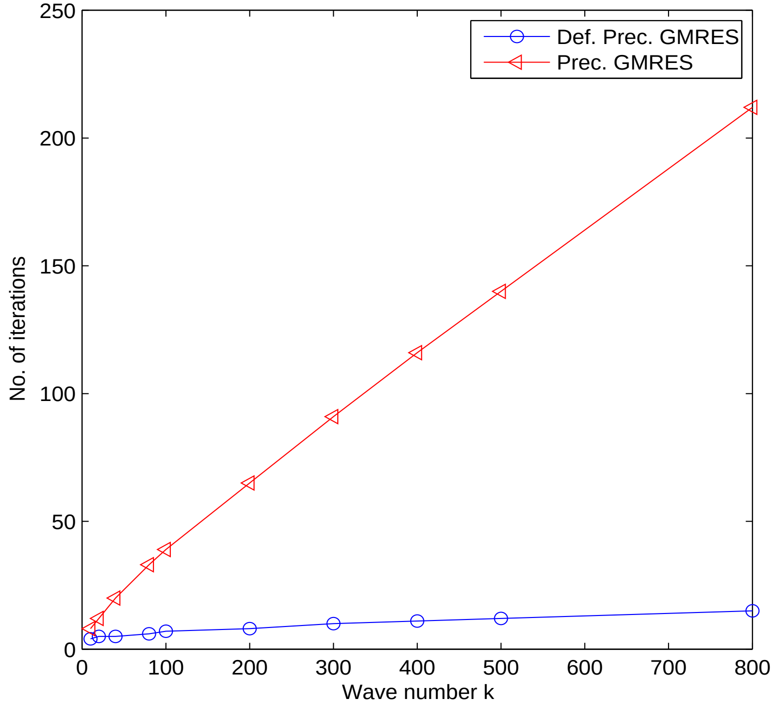
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Numerical results

Number of GMRES iterations with/without deflation to solve a Wedge problem. Shifts in the preconditioner are $(1, 0.5)$

Grid	$freq = 10$	$freq = 20$	$freq = 30$	$freq = 40$	$freq = 50$
74×124	**7/33**	20/60	79/95	267/156	490/292
148×248	5/33	**9/57**	17/83	42/112	105/144
232×386	5/33	7/57	**10/81**	25/108	18/129
300×500	4/33	6/57	8/81	**12/105**	18/129
374×624	4/33	5/57	7/80	9/104	**13/128**

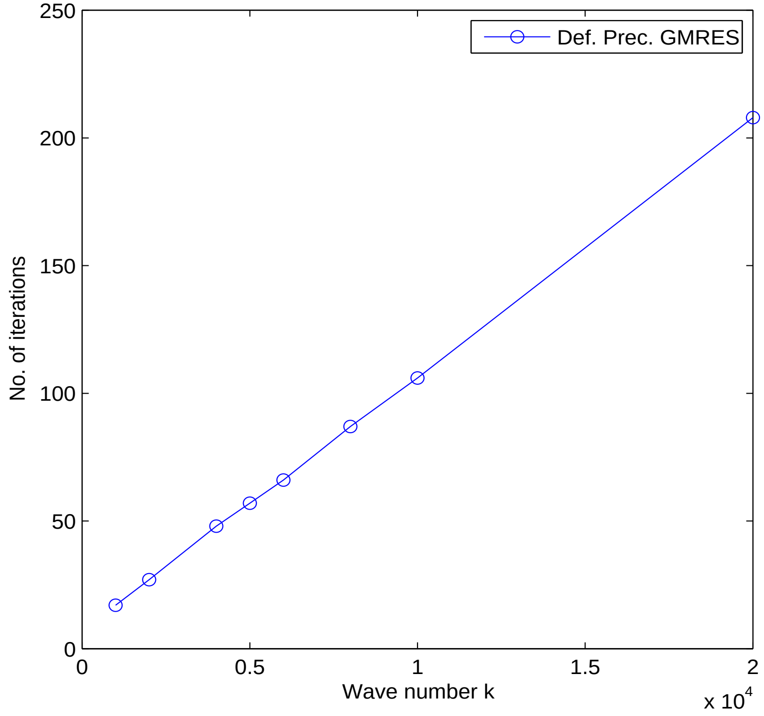
Numerical results



- Number of GMRES iterations for the 1D Helmholtz equation

$$10 \leq k \leq 800$$

Numerical results



Number of GMRES iterations for the 1D Helmholtz equation

$$1000 \leq k \leq 20000$$

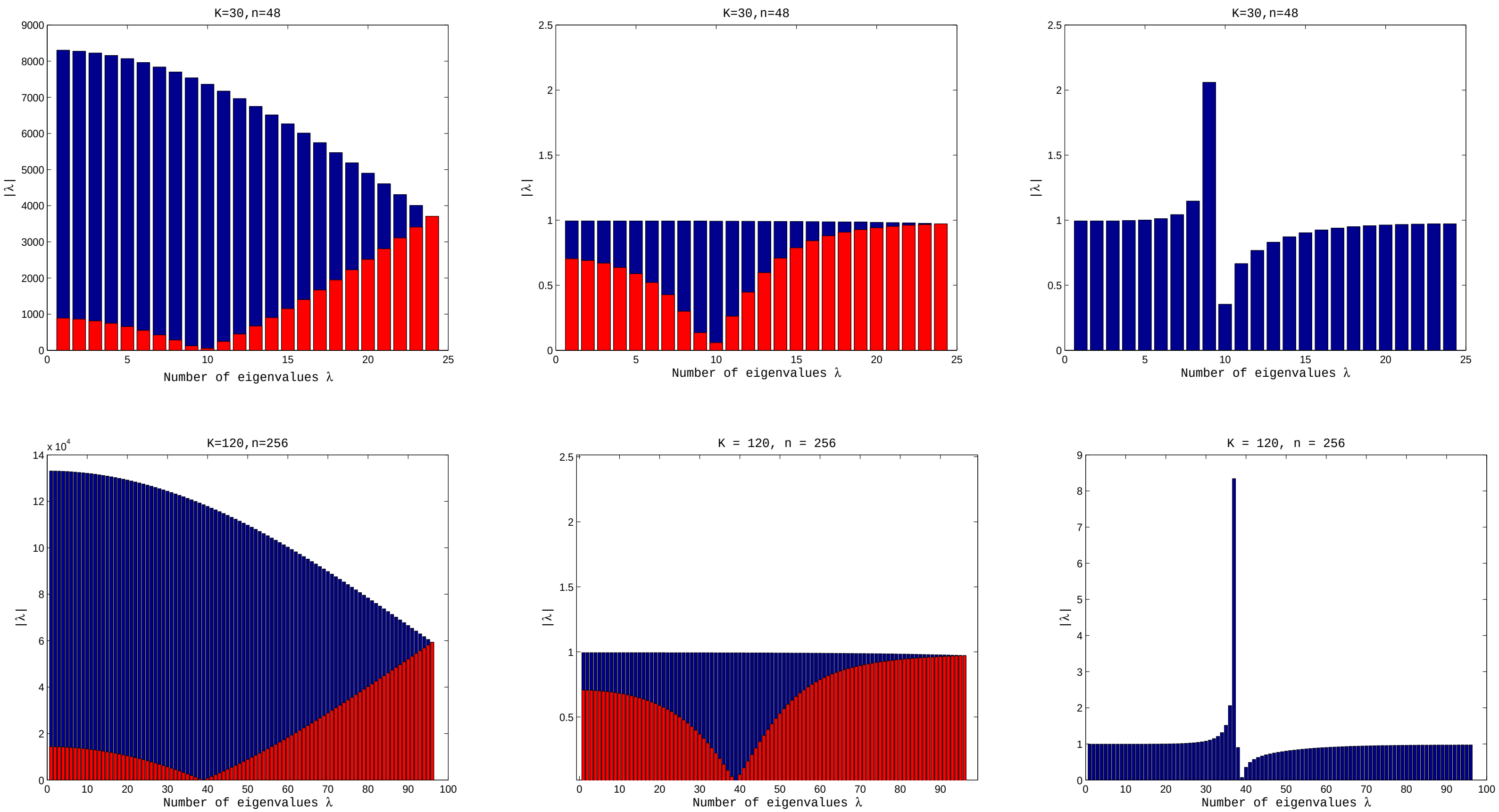
Numerical results

Number of GMRES outer-iterations in multilevel algorithm.
$(\beta_1, \beta_2) = (1, 0.5)$ $kh = .3125$ or 20 gp/wl
and MG Vcycle(1,1) for SLP

Grid	$k = 10$	$k = 20$	$k = 40$	$k = 80$	$k = 160$
MLMGV(4,2,1)	9	11	16	27	100+
MLMGV(6,2,1)	9	10	14	21	47
MLMGV(8,2,1)	9	10	13	20	38
MLMGV(8,3,2)	9	10	13	19	37

Fourier Analysis

Spectrum of A, $M^{-1}A$ and $PM^{-1}A$ (from left to right) in bar-graph.



Conclusions

- Parameter (in)dependent scheme.
- Numerical results confirm analysis.
- Without deflation, when imaginary shift is increased in SLP, spectrum remains bounded above 1, but lower part moves to zero.
- Flexibility to increase imaginary shift, when deflation is combined with SLP.
- Further research Multilevel scheme, applying similarly for coarse problem in deflation. Questions: gain in CPU time? why not scalable? ...

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