Physics-based preconditioners for large-scale subsurface flow simulation.

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SPE 10

Single-phase flow, grid size 60 x 220 x 85 grid cells.

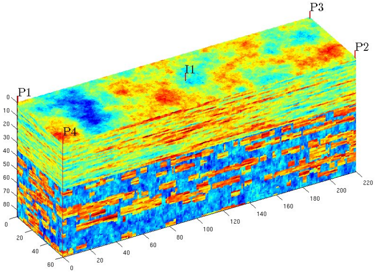

Figure : Permeability field SPE10.

Method	Number or iterations
ICCG	1029
DICCG	

Table : Number of iterations for the SPE 10 benchmark for the ICCG and DICCG methods, tolerance of 10^{-11}.

SPE 10

Single-phase flow, grid size 60 x 220 x 85 grid cells.

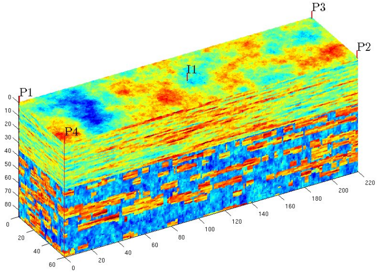

Figure : Permeability field SPE10.

Method	Number or iterations
ICCG	1029
DICCG	1

Table : Number of iterations for the SPE 10 benchmark (85 layers) for the ICCG and DICCG methods, tolerance of 10^{-7}.

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Optimal Control

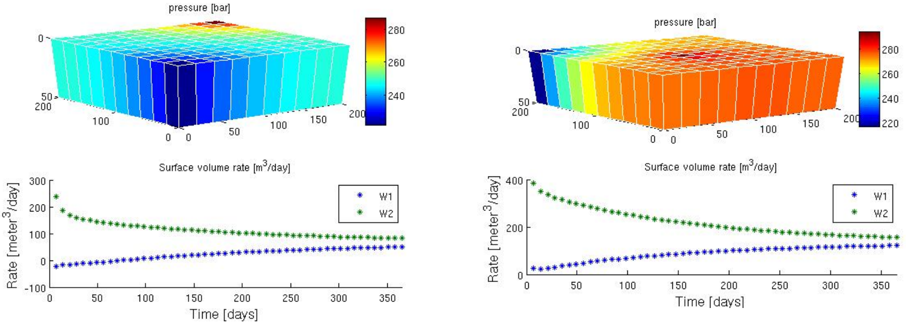


Figure : Optimal Control[1].

[1] MRST [4]

Problem Definition

Reservoir Simulation

Single-phase flow through porous media [5]

Darcy's law + mass balance equation

$$-\nabla \cdot \left[\frac{\alpha\rho}{\mu} \vec{\mathbf{K}} (\nabla \mathbf{p} - \rho g \nabla d) \right] + \alpha\rho\phi c_t \frac{\partial \mathbf{p}}{\partial t} - \alpha\rho \mathbf{q} = 0.$$

$$c_t = (c_l + c_r),$$

α a geometric factor
ρ fluid density
μ fluid viscosity
$\mathbf{p}$ pressure
$\vec{\mathbf{K}}$ rock permeability

g gravity
d depth
ϕ rock porosity
q sources
c_r rock compressibility
c_l liquid compressibility

Problem Definition

Discretization

2D case, isotropic permeability, small rock and fluid compressibilities, uniform reservoir thickness and no gravity forces.

$$-\frac{h}{\mu}\frac{\partial}{\partial x}\left(k\frac{\partial \mathbf{p}}{\partial x}\right)-\frac{h}{\mu}\frac{\partial}{\partial y}\left(k\frac{\partial \mathbf{p}}{\partial y}\right)-\frac{h}{\mu}\frac{\partial}{\partial z}\left(k\frac{\partial \mathbf{p}}{\partial z}\right)+h\phi_0 c_t\frac{\partial \mathbf{p}}{\partial t}-h\mathbf{q}=0.$$

$$\boxed{\mathcal{V}\dot{\mathbf{p}}+\mathcal{T}\mathbf{p}=\mathbf{q}.}$$

Accumulation matrix

$$\mathcal{V}=Vc_t\phi_0\mathcal{I},$$

$$V=h\Delta x\Delta y\Delta z.$$

Transmissibility matrix

$$\mathcal{T}_{i-\frac{1}{2},j}=\frac{\Delta y}{\Delta x}\frac{h}{\mu}k_{i-\frac{1}{2},j},$$

$$k_{i-\frac{1}{2},j}=\frac{2}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}.$$

Problem Definition

Incompressible model

$$\boxed{\mathcal{T}\mathbf{p} = \mathbf{q}.}$$

Properties of $\mathcal{T}$

Eigenvalues

$$\mathcal{T}\mathbf{p} = \lambda\mathbf{p}$$

Condition number of a SPD matrix.

$$\kappa_2(\mathcal{T}) = \frac{\lambda_{max}(\mathcal{T})}{\lambda_{min}(\mathcal{T})}$$

$\mathbf{q}$: sources or wells in the reservoir.

Peaceman well model

$$\mathbf{q} = -J_{well}(\mathbf{p} - \mathbf{p}_{well})$$

J_{well} is the well index, negative sign is a production well.

Conjugate Gradient Method (CG)

Successive approximations to obtain a more accurate solution $\mathbf{x}$ [6]

$$\mathcal{A}\mathbf{x} = \mathbf{b},$$

$$\mathbf{x}^0, \qquad \text{initial guess}$$

$$\vdots$$

$$\mathbf{x}^k = \mathbf{x}^{k-1} + \mathcal{M}^{-1}\mathbf{r}^{k-1}, \qquad \mathbf{r}^k = \mathbf{b} - \mathcal{A}\mathbf{x}^{k-1}.$$

$$min_{\mathbf{x}^k \in \kappa_k(\mathcal{A},\mathbf{r}^0)} ||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}}, \qquad ||\mathbf{x}||_{\mathcal{A}} = \sqrt{\mathbf{x}^T \mathcal{A}\mathbf{x}}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{A})} + 1} \right)^{k+1}.$$

PCG (ICCG)

Preconditioning

Improve the spectrum of $\mathcal{A}$.

$$\mathcal{M}^{-1}\mathcal{A}\mathbf{x} = \mathcal{M}^{-1}\mathbf{b}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} + 1} \right)^{k+1},$$

$$\kappa(\mathcal{M}^{-1}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

Cholesky Decomposition
If $\mathcal{A} \in \mathcal{R}^{n \times n}$ is *SPD*,

$$\mathcal{A} = \mathcal{L}\mathcal{L}^T$$

IC(0)
Let $a_{i,j} \in \mathcal{A}$ and $l_{i,j} \in \mathcal{L}^*$, $\mathcal{L}^*$ the matrix from the Cholesky decomposition, such that $l_{i,j} = 0$ if $a_{i,j} = 0$.

DPCG

Deflation

$$\mathcal{P} = \mathcal{I} - \mathcal{A}\mathcal{Q}, \qquad \mathcal{P} \in \mathbb{R}^{n \times n}, \qquad \mathcal{Q} \in \mathbb{R}^{n \times n},$$
$$\mathcal{Q} = \mathcal{Z}\mathcal{E}^{-1}\mathcal{Z}^{T}, \qquad \mathcal{Z} \in \mathbb{R}^{n \times k}, \qquad \mathcal{E} \in \mathbb{R}^{k \times k},$$
$$\mathcal{E} = \mathcal{Z}^{T}\mathcal{A}\mathcal{Z} \text{ (Tang 2008, [7]).}$$

Convergence

Deflated system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{P}\mathcal{A})} + 1} \right)^{k+1}.$$

Deflated and preconditioned system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} + 1} \right)^{k+1}.$$

$$\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A}) \leq \kappa_{eff}(\mathcal{P}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

Deflation vectors

Recycling deflation (Clemens 2004, [8]).

$$\mathcal{Z} = [\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^{q-1}],$$

x^i's are solutions of the system.
Multigrid and multilevel (Tang 2009, [9]).
The matrices $\mathcal{Z}$ and $\mathcal{Z}^T$ are the restriction and prolongation matrices of multigrid methods.
Subdomain deflation (Vuik 1999,[10]).

Proposal

- Use solution of the system with various well configurations as deflation vectors (Recycling deflation).
- Use as deflation vectors the basis obtained from Proper Orthogonal Decomposition (POD).

Deflation vectors

Lemma 1. Let $\mathcal{A} \in \mathbb{R}^{n \times n}$ be a non-singular matrix, such that

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

and $\mathbf{x}_i, \mathbf{b}_i \in \mathbb{R}^n$, $i = 1, ..., m$, $\mathbf{b}_i$ are linearly independent ($l.i.$) such that:

$$\mathcal{A}\mathbf{x}_i = \mathbf{b}_i. \tag{2}$$

The following equivalence holds

$$\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \qquad \Leftrightarrow \qquad \mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i. \tag{3}$$

Proof $\Rightarrow$ Substituting $\mathbf{x}$ from (3) into $\mathcal{A}\mathbf{x} = \mathbf{b}$, and using linearity of $\mathcal{A}$ and(2):

$$\begin{aligned} \mathcal{A}\mathbf{x} &= \sum_{i=1}^{m} \mathcal{A} c_i \mathbf{x}_i = \mathcal{A}(c_1\mathbf{x}_1 + ... + c_m\mathbf{x}_m) \\ &= \mathcal{A}c_1\mathbf{x}_1 + ... + \mathcal{A}c_m\mathbf{x}_m = c_1\mathbf{b}_1 + ... + c_m\mathbf{b}_m = \sum_{i=1}^{m} c_i\mathbf{b}_i. \end{aligned} \tag{4}$$

Similar proof for $\Leftarrow$

Deflation vectors

Lemma 2. If the the deflation matrix $\mathcal{Z}$ is constructed with a set of m vectors

$$\mathcal{Z} = \begin{bmatrix} \mathbf{x}_1 & \dots & \dots & \mathbf{x}_m \end{bmatrix},$$

such that $\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i$, with $\mathbf{x}_i$ *l.i.*, then the solution of system $\mathcal{A}\mathbf{x} = \mathbf{b}$ is achieved within one iteration of DCG.

Proof.

The relation between $\hat{\mathbf{x}}$ and $\mathbf{x}$ is given as:

$$\mathbf{x} = \mathcal{Q}\mathbf{b} + \mathcal{P}^T \hat{\mathbf{x}}. \tag{5}$$

For the first term $\mathcal{Q}\mathbf{b}$, taking $\mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i$ we have:

$$\mathcal{Q}\mathbf{b} = \mathcal{Z}\mathcal{E}^{-1}\mathcal{Z}^T \left(\sum_{i=1}^{m} c_i \mathbf{b}_i \right) = \mathcal{Z}(\mathcal{Z}^T \mathcal{A} \mathcal{Z})^{-1} \mathcal{Z}^T \left(\sum_{i=1}^{m} c_i \mathcal{A} \mathbf{x}_i \right) = \qquad \text{Lemma 1}$$

$$= \mathcal{Z}(\mathcal{Z}^T \mathcal{A} \mathcal{Z})^{-1} \mathcal{Z}^T (\mathcal{A}\mathbf{x}_1 c_1 + \dots + \mathcal{A}\mathbf{x}_m c_m) = \mathcal{Z}(\mathcal{Z}^T \mathcal{A} \mathcal{Z})^{-1} \mathcal{Z}^T \mathcal{A} \mathcal{Z} \mathbf{c}$$

$$= \mathcal{Z}\mathbf{c} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{x}_3 + c_4 \mathbf{x}_4 + c_5 \mathbf{x}_5 = \sum_{i=1}^{m} c_i \mathbf{x}_i = \mathbf{x}.$$

Proper Orthogonal Decomposition (POD)

POD: find an 'optimal' basis for a given data set (Markovinović 2009 [1], Astrid 2011, [2])

$$\Phi = [\phi_1, \phi_2, ...\phi_l] \in \mathbb{R}^{n \times l}$$

ϕ_i, basis functions.

- Get the snapshots
$$\mathcal{X} = [\mathbf{x}_1, \mathbf{x}_2, ...\mathbf{x}_m].$$
- l eigenvectors of $\mathcal{R}$ satisfying:
$\frac{\sum_{j=1}^{l} \lambda_j}{\sum_{j=1}^{m} \lambda_j} \leq \alpha, \qquad 0 < \alpha \leq 1.$

$$\mathcal{R} := \frac{1}{m}\mathcal{X}\mathcal{X}^T \equiv \frac{1}{m}\sum_{i=1}^{m} \mathbf{x}_i \mathbf{x}_i^T.$$

Numerical experiments

Case 1. Heterogeneous permeability.
The experiments were performed for single-phase flow, with the following characteristics:

- Grid size $nx \times ny$ grid cells, $nx = ny = 64$.
- Permeability 1 mD.
- W1 = W2 = W3 = W4 = -1 bars.
- W5 = +4 bars.
- Neumann boundary conditions.

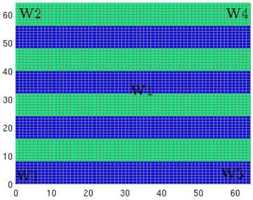


Figure : Heterogeneous permeability layers.

Numerical experiments (Heterogeneous permeability)

Snapshots

$\mathbf{z}_1$: W1 = 0 bars, W2 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_2$: W2 = 0 bars, W1 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_3$: W3 = 0 bars, W1 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_4$: W4 = 0 bars, W1 = W2 = W3 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_5$: W1 = W2 = W3 = W4 = -1 bars, W5 = b5 = +4 bars.

Results

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
ICCG	90	131	65*	64*
$DICCG_4$	1	1	1*	1*
$DICCG_5$	1	500*	500*	500*

Table : Number of iterations for different contrast in the permeability of the layers ($\sigma_1 = 1mD$) for the ICCG and DICCG methods, tolerance of 10^{-11}, snapshots 10^{-11}. $DICCG_4$ is the method with 4 deflation vectors and $DICCG_5$ is the method with 5 deflation vectors.

Numerical experiments (Heterogeneous permeability)

Condition number of an SPD matrix.

$$\kappa_2(\mathcal{A}) = \frac{\lambda_{max}(\mathcal{A})}{\lambda_{min}(\mathcal{A})}$$

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
$\kappa(A)$	2.6×10^3	$2.4{\times}10^5$	2.4×10^7	2.4×10^9
$\kappa(M^{-1}A)$	206.7	8.3×10^3	8.3×10^5	8.3×10^7
$\kappa_{eff}(M^{-1}PA)$	83.27	6×10^3	1×10^6	6×10^7

Table : Condition number for various permeability contrasts between the layers, grid size of 32 x 32, $\sigma_1 = 1mD$.

Numerical experiments (Heterogeneous permeability)

Relative error

$$e = \frac{||\mathbf{x} - \mathbf{x}^k||_2}{||\mathbf{x}||_2} \leq \kappa_2(A)\epsilon,$$

with $\mathbf{x}$ the true solution and $\mathbf{x}^k$ the approximation

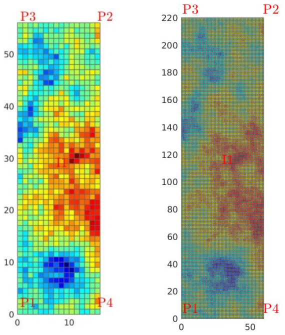
Taking $e = 10^{-7}$.

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
$tol = \frac{e}{\kappa_2(M^{-1}A)} = \frac{10^{-7}}{\kappa_2(M^{-1}A)}$	5×10^{-9}	1×10^{-10}	10^{-12}	1×10^{-14}
$tol = \frac{e}{\kappa_{eff}(M^{-1}PA)} = \frac{10^{-7}}{\kappa_{eff}(M^{-1}PA)}$	1×10^{-8}	2×10^{-10}	10^{-12}	2×10^{-14}

Table : Tolerance needed for various permeability contrast between the layers, grid size of 32 x 32, $\sigma_1 = 1mD$, for an error of $e = 10^{-7}$.

Numerical experiments (SPE 10)

SPE 10 model, 2nd layer



(a) 16 x 56 (b) 60 x 220

Figure : Permeability field, 16 x 56 and 60 x 220 grid cells.

Grid size	16 x 56	30 x 110	46 x 166	60 x 220
Contrast $(\times 10^7)$	1.04	2.52	2.6	2.8

Table : Contrast in permeability for different grid sizes $(\sigma_{max}/\sigma_{min})$.

Condition number	value
$\kappa(A)$	2.2×10^6
$\kappa(M^{-1}A)$	377
$\kappa_{eff}(M^{-1}PA)$	82.7

Table : Table with the condition number of the SPE10 model, grid size of 16 x 56.

SPE 10 model, 2nd layer

Tol (snap-shots)	Method	16 x 56	30 x 110	46 x 166	60 x 220
	ICCG	34	73	126	159
10^{-1}	$DICCG_4$	33	72	125	158
	$DICCG_5$	500*	500*	500*	500*
10^{-3}	$DICCG_4$	18	38	123	151
	$DICCG_5$	18	35	123	150
10^{-5}	$DICCG_4$	11	21	27	55
	$DICCG_5$	9	22	23	54
10^{-7}	$DICCG_4$	1	1	1	1
	$DICCG_5$	1	1	1	1

Table : Number of iterations for ICCG and DICCG, diverse tolerance for the snapshots, different grid sizes. $DICCG_4$ is computed with 4 deflation vectors, $DICCG_5$ with 5.

Numerical experiments

SPE 10 model, 85 layers

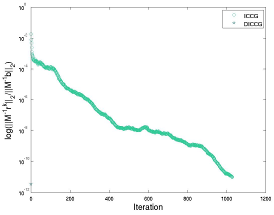


Figure : Convergence plot ICCG and DICCG.

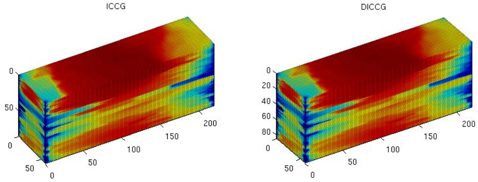


Figure : Solution ICCG and DICCG.

Method	Number or iterations
ICCG	1029
DICCG	1

Table : Number of iterations ICCG and DICCG, relative tolerance 10^{-11}.

Conclusions

- Solution is reached in 1 iteration for DICCG method.
- Number of iterations for the DICCG method does not depend on the contrast between the coefficients (Heterogeneous permeability example).
- Number of iterations for the DICCG method does not depend on the grid size (SPE 10 example).
- The choice of deflation vectors is important for a good performance of DICCG.

Future work.

- Study the computation time of DICCG.
- Work with compressible models.
- Other snapshots?.
- Use of POD for the selection of deflation vectors.

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