Iterative Helmholtz Solvers

Multigrid for Helmholtz: Towards Convergence

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March 31, 2021

Aim and Impact

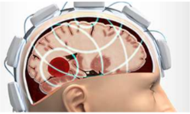
- Joint-work with Professor Kees Vuik
- Contribute to broad research on Helmholtz solvers
- Understand convergence/divergence behavior
- Improve convergence properties

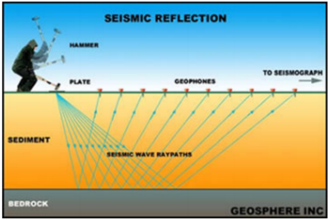
Introduction - The Helmholtz Equation

- Inhomogeneous Helmholtz equation + BC's

$$(-\nabla^2 - k^2)\, u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega \subseteq \mathbb{R}^n$$

- k is the wave number: $k = \frac{2\pi}{\lambda}$
- Practical applications in seismic and medical imaging





Introduction - Numerical Model

- Start with analytical 1D model problem

$$-\frac{d^2u}{dx^2} - k^2\,u = \delta(x - \frac{1}{2}),$$
$$u(0) = 0, u(1) = 0,$$
$$x \in \Omega = [0, 1] \subseteq \mathbb{R},$$

- Discretization using second-order FD with at least 10 gpw
- We obtain a linear system $A\hat{u} = f$

$$A = \frac{1}{h^2}\text{tridiag}[-1 \;\; 2 - (kh)^2 \;\; -1],$$

- A is real, symmetric, normal, indefinite and sparse
- Using Radiation BC's A becomes non-Hermitian

Multigrid - Challenges

- Oscillations cause ineffective coarse grid (phase-lead)
- Low-frequency error not eliminated
- Near-zero eigenvalues
- Some remedies so far:

Wave-ray multigrid	- constant k
Dispersion correction	- constant k + coarse resolution restriction
GMRES smoothing	- level-dep. + 'manually' optimized param.
Complex stretched grids	- level-dep.

- Our approach: h.o. intergrid transfer operators
- Background from work on deflation

Background - Two-Level Deflation

- Projection principle: solve $PAu = Pf$

$$\tilde{P} = AQ \text{ where } Q = ZE^{-1}Z^T \text{ and } E = Z^TAZ,$$
$$P = I - \tilde{P},\ Z \in \mathbb{R}^{m \times n},\ m < n$$

- Columns of Z span deflation subspace
- Inter-grid vectors from multigrid as deflation vectors
- Apply P as a preconditioner: solve $PA\hat{u} = Pf$

Background - Two-Level Deflation

- Deflation space spanned by linear approximation basis vectors
- Transfer coarse-fine grid ⇒ interpolation error
- Measure effect by projection error E

$$E(kh) = \|(I - P)\phi_{j_{\min},h}\|^2,$$

$$P = Z(Z^T Z)^{-1} Z^T$$

Figure: Restricted & interpolated eigenvectors (left $kh = 0.625$, right $k^3h^2 = 0.625$

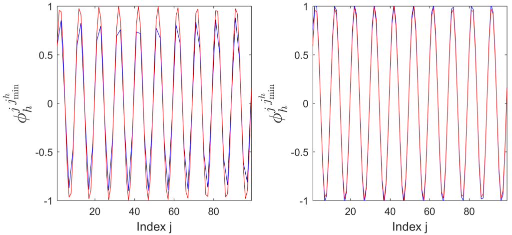


Table: Projection error

k	$E(0.625)$	$E(0.3125)$
10^2	0.8818	0.1006
10^3	9.2941	1.0062
10^4	92.5772	10.0113
10^5	926.135	100.1382
10^6	9261.7129	1001.3818

Background - Our Approach

- Higher-order deflation vectors
- Rational quadratic Bezier curve $\Rightarrow$ one control-point
- This scheme results in close to wave number independent convergence
- But what about multigrid?

Multigrid - 1D

- Constant wave number using Dirichlet BC
- Construct two-level V(1,1)-cycle + weighted Jacobi smoothing

Table: Two-grid spectral radius

k	Quadratic Bezier		Linear	
	$kh = 0.625$	$kh = 0.3125$	$kh = 0.625$	$kh = 0.3125$
50	0.2436	0.2852	1.290	0.9217
100	0.2441	0.2076	3.325	1.0225
250	0.2443	0.1538	5.4108	21.5327
500	0.2443	0.1354	15.5047	21.5327
1000	0.2443	0.1350	27.7478	21.5327

- As expected: original multigrid setup (linear) diverges
- H.o. scheme gives spectral radius *strictly* < 1
- Analogous to projection error *strictly* < 1 for deflation!

Multigrid - 1D

- Constant wave number using Dirichlet BC
- weighted Jacobi smoothing

Table: Number of iterations

k	Two-level Deflation		Two-grid V(1,1)-cycle	
	$kh = 0.625$	$kh = 0.3125$	$kh = 0.625$	$kh = 0.3125$
10^2	9	9	10	8
10^3	9	9	10	8
10^4	9	9	10	9
10^5	9	9	11	12

- Both schemes close to wave number independent convergence

Multigrid - 2D

- Constant wave number using Sommerfeld BC
- Construct two-level V(1,1)-cycle

k	$\omega-$Jacobi		Gaus-Seidel	
	$kh = 0.625$	$kh = 0.3125$	$kh = 0.625$	$kh = 0.3125$
50	14	14	6	5
100	14	14	6	5
250	14	14	6	5
500	14	14	6	5

- Both cases wave number independence
- Still exact solve on second-level $\Rightarrow$ memory constraints
- Can we create a deeper V-cycle?

Multigrid - 2D

- Constant wave number using Sommerfeld BC
- Three-grid cycle with $kh_{coarsest} = 2.5 \approx \frac{2\pi}{2.5}$

Figure: V-cycle

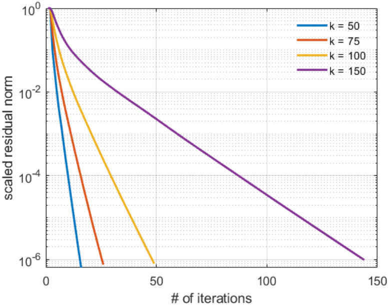
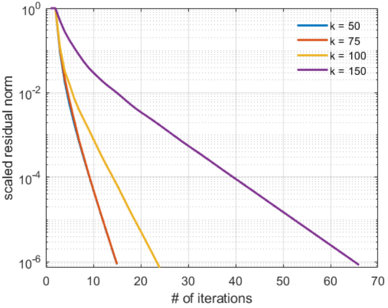


Figure: F-cycle



- Convergence no longer wave number independent
- Deeper cycle diverges
- Remedy: use coarsening on CSL (S. Cools)

Multigrid - 2D

- Constant wave number using Sommerfeld BC

Table: Number of V- ($\gamma = 1$) and W-cycles ($\gamma = 2$) for constant k using a stopping tolerance of 10^{-5} and ν denotes the number of ω-Jacobi smoothing steps.

	$k = 50$		$k = 100$		$k = 150$		$k = 200$		$k = 250$	
	$N = 6724$ $N_D = 8$		$N = 26244$ $N_D = 8$		$N = 57600$ $N_D = 4$		$N = 102400$ $N_D = 8$		$N = 160000$ $N_D = 4$	
γ	1	2	1	2	1	2	1	2	1	2
$\nu = 4$	58	58	104	108	155	159	209	213	267	271
$\nu = 5$	58	58	104	104	150	166	194	229	238	287
$\nu = 6$	55	58	99	102	139	167	183	222	226	283
$\nu = 7$	53	60	97	101	136	163	179	219	221	280
$\nu = 8$	53	60	95	104	131	161	178	212	218	277

- Coarsening + $w-$Jacobi smoothing on CSL (shift = 0.7)
- No level-dependent parameters!
- Linear interpolation diverges ($k = 50, \gamma = 1$)
- What about heterogeneous problems?

Multigrid - 2D

Figure: $k(x, y)$

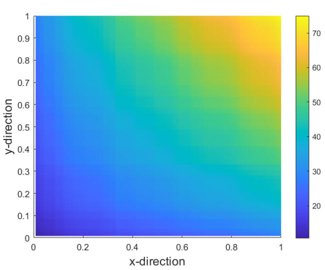


Figure: $u(x, y)$

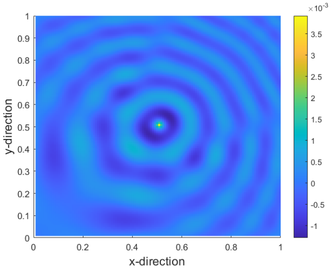


Table: Number of V- ($\gamma = 1$) and W-cycles ($\gamma = 2$) with tol 10^{-5}. ν denotes the number of ω-Jacobi smoothing steps.

	$(k_1, k_2) = (10, 50)$		$(k_1, k_2) = (10, 75)$	
γ	1	2	1	2
$\nu = 4$	69	66	98	90
$\nu = 5$	66	66	90	90
$\nu = 6$	68	66	124	96
$\nu = 7$	71	67	145	95
$\nu = 8$	74	69	159	96

Multigrid - 2D

Figure: $k(x,y)$

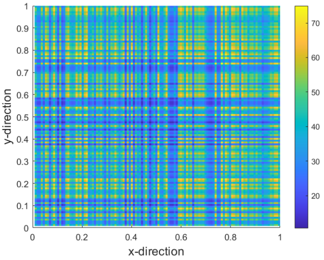


Figure: $u(x,y)$

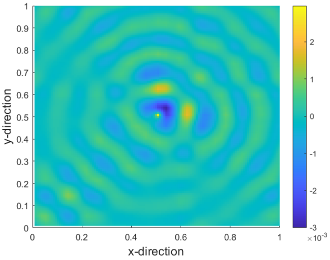


Table: Number of V- ($\gamma = 1$) and W-cycles ($\gamma = 2$) with tol 10^{-5}. ν denotes the number of ω-Jacobi smoothing steps.

	$(k_1, k_2) = (10, 50)$		$(k_1, k_2) = (10, 75)$	
γ	1	2	1	2
$\nu = 4$	123	108	139	128
$\nu = 5$	112	110	129	128
$\nu = 6$	112	114	128	130
$\nu = 7$	116	116	131	133
$\nu = 8$	123	123	135	137

Multigrid - Status-quo

- Current setup works for non-constant wavenumbers
- No level-dependent parameters
- Convergence using standard $w-$Jacobi smoothing
- Full coarsening until size coarse system $< 10 \times 10$
- Iteration number grows with wavenumber
- Can we reduce number of iterations using GMRES smoothing? (H. Elman, S. Cools)

Multigrid - 2D

- Constant wave number using Sommerfeld BC

Table: Number of V- ($\gamma = 1$) and W-cycles ($\gamma = 2$) for constant k using tol. 10^{-5}. ν denotes the number of GMRES(3) smoothing steps.

	$k = 50$		$k = 100$		$k = 150$		$k = 200$		$k = 250$	
	$N = 6724$ $N_D = 8$		$N = 26244$ $N_D = 8$		$N = 57600$ $N_D = 4$		$N = 102400$ $N_D = 8$		$N = 160000$ $N_D = 4$	
γ	1	2	1	2	1	2	1	2	1	2
$\nu = 1$	37	36	68	67	99	98	132	131	162	161
$\nu = 2$	29	29	53	53	78	78	104	104	128	128
$\nu = 3$	24	24	45	45	67	67	89	89	112	112
$\nu = 4$	22	22	40	40	59	59	78	78	98	98
$\nu = 5$	20	20	36	36	53	53	71	71	88	88

- Coarsening + GMRES(3) smoothing on CSL (shift = 0.7)
- Number of iterations scale linearly with k
- Linear interpolation 199 iterations ($k = 50, \gamma = 1$)

Multigrid - 2D

- Constant wave number using Sommerfeld BC

Table: Number of V- ($\gamma = 1$) and W-cycles ($\gamma = 2$) for constant k using tol. 10^{-5}. ν denotes the number of GMRES(3) smoothing steps.

	$k = 50$		$k = 100$		$k = 150$		$k = 200$		$k = 250$	
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γ	1	2	1	2	1	2	1	2	1	2
$\nu = 1$	14	7	24	10	39	19	51	24	64	29
$\nu = 2$	8	5	13	7	22	10	28	13	34	16
$\nu = 3$	6	5	10	6	16	9	20	10	24	12
$\nu = 4$	6	5	8	5	12	7	15	9	18	10
$\nu = 5$	5	5	7	5	11	7	13	8	15	9

- Coarsening + GMRES(3) smoothing on CSL (shift $= k^{-1}$)
- Iteration count with $\gamma = 2$ close to k-independent
- Linear interpolation 248 iterations ($k = 50, \gamma = 1$)

Conclusion

- H.o. intergrid vectors for deflation
- Apply similar approach to multigrid
- Converges until coarse system negative definite
- Fix using CSL for coarsening and smoothing
- Result: level-independent convergent V-cycle
- No restrictions to coarse grid resolution
- Some challenges remain:
  - What about fully Dirichlet BC's?
  - For better iteration numbers $\Rightarrow$ GMRES smoothing

What's next?

- Future work on h.o. intergrid operators
- Assess quality of different smoothers
- Provide analysis and theory
- Investigate more heterogeneous and 3D models
- Investigate performance as a preconditioner

References

- Upcoming articles: multilevel deflation and convergent multigrid methods for the Helmholtz equation.
- Further reading

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