Robust and Fast Preconditioners for Porous Media Flow

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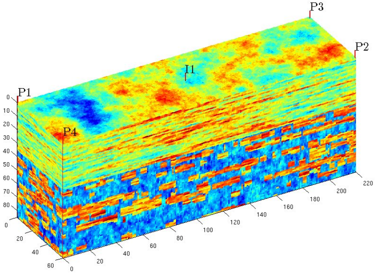
SPE 10

Single-phase flow, grid size 60 x 220 x 85 grid cells.

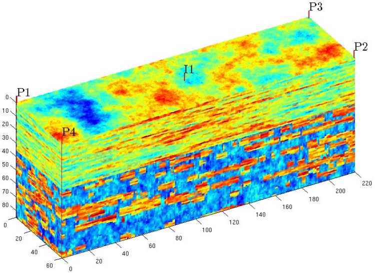


Method	Number of iterations
ICCG	1011
DICCG	

Table : Number of iterations for the SPE 10 benchmark (85 layers) for the ICCG and DICCG methods, tolerance of 10^{-7}.

SPE 10

Single-phase flow, grid size 60 x 220 x 85 grid cells.



Method	Number of iterations
ICCG	1011
DICCG	2

Table : Number of iterations for the SPE 10 benchmark (85 layers) for the ICCG and DICCG methods, tolerance of 10^{-7}.

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Problem Definition

Reservoir Simulation

Single-phase flow through a porous media [1]

Darcy's law + mass balance equation

$$-\nabla \cdot \left[\frac{\alpha\rho}{\mu}\vec{\mathbf{K}}(\nabla\mathbf{p} - \rho g \nabla d)\right] + \alpha\rho\phi c_t \frac{\partial \mathbf{p}}{\partial t} - \alpha\rho\mathbf{q} = 0.$$

$$c_t = (c_l + c_r),$$

α a geometric factor
ρ fluid density
μ fluid viscosity
$\mathbf{p}$ pressure
$\vec{\mathbf{K}}$ rock permeability

g gravity
d depth
ϕ rock porosity
$\mathbf{q}$ sources
c_r rock compressibility
c_l liquid compressibility

Problem Definition

Discretization

2D case, isotropic permeability, small rock and fluid compressibilities, uniform reservoir thickness and no gravity forces.

$$-\frac{h}{\mu}\frac{\partial}{\partial x}\left(k\frac{\partial \mathbf{p}}{\partial x}\right)-\frac{h}{\mu}\frac{\partial}{\partial y}\left(k\frac{\partial \mathbf{p}}{\partial y}\right)-\frac{h}{\mu}\frac{\partial}{\partial z}\left(k\frac{\partial \mathbf{p}}{\partial z}\right)+h\phi_0 c_t\frac{\partial \mathbf{p}}{\partial t}-h\mathbf{q}=0.$$

$$\boxed{\mathcal{V}\dot{\mathbf{p}}+\mathcal{T}\mathbf{p}=\mathbf{q}.}$$

$\mathbf{q}$: sources or wells in the reservoir, Peaceman well model, $\mathcal{I}_{well}$ is the well index

$$\mathbf{q}=-\mathcal{I}_{well}(\mathbf{p}-\mathbf{p}_{well})$$

Accumulation matrix

$$\mathcal{V}=Vc_t\phi_0\mathcal{I},$$

$$V=h\Delta x\Delta y\Delta z.$$

Transmissibility matrix

$$\mathcal{T}_{i-\frac{1}{2},j,l}=\frac{\Delta y}{\Delta x\Delta z}\frac{h}{\mu}k_{i-\frac{1}{2},j,l},$$

$$k_{i-\frac{1}{2},j}=\frac{2}{\frac{1}{k_{i-1,j,l}}+\frac{1}{k_{i,j,l}}}.$$

Problem Definition

Incompressible model

$$\mathcal{T}\mathbf{p} = \mathbf{q}.$$

Compressible model

$$\mathcal{V}^{n+1}\frac{(\mathbf{p}^{n+1} - \mathbf{p}^{n})}{\Delta t^{n}} + \mathcal{T}^{n+1}\mathbf{p}^{n+1} = \mathbf{q}^{n+1}.$$

Or:

$$\mathcal{F}(\mathbf{p}^{n+1}; \mathbf{p}^{n}) = 0. \qquad (1)$$

Newton-Raphson

Using Newton-Raphson (NR) method, the system for the $(k+1)$-th NR iteration is:

$$\mathcal{J}(\mathbf{p}^k)\delta\mathbf{p}^{k+1} = -\mathcal{F}(\mathbf{p}^k;\mathbf{p}^n), \qquad \mathbf{p}^{k+1} = \mathbf{p}^k + \delta\mathbf{p}^{k+1},$$

where $\mathcal{J}(\mathbf{p}^k) = \frac{\partial\mathcal{F}(\mathbf{p}^k;\mathbf{p}^n)}{\partial\mathbf{p}^k}$ is the Jacobian matrix, and $\delta\mathbf{p}^{k+1}$ is the NR update at iteration step $k+1$.

$$\mathcal{J}(\mathbf{p}^k)\delta\mathbf{p}^{k+1} = \mathbf{b}(\mathbf{p}^k). \tag{2}$$

Conjugate Gradient Method (CG)

Successive approximations to obtain a more accurate solution $\mathbf{x}$ [2]

$$\mathcal{A}\mathbf{x} = \mathbf{b},$$

$$\mathbf{x}^0, \qquad \text{initial guess} \qquad \mathbf{r}^k = \mathbf{b} - \mathcal{A}\mathbf{x}^{k-1}.$$

$$min_{\mathbf{x}^k \in \kappa_k(\mathcal{A}, \mathbf{r}^0)} ||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}}, \qquad ||\mathbf{x}||_{\mathcal{A}} = \sqrt{\mathbf{x}^T \mathcal{A}\mathbf{x}}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{A})} + 1} \right)^k.$$

Preconditioning

Improve the spectrum of $\mathcal{A}$.

$$\mathcal{M}^{-1}\mathcal{A}\mathbf{x} = \mathcal{M}^{-1}\mathbf{b}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} + 1} \right)^k, \qquad \kappa(\mathcal{M}^{-1}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

DPCG

Deflation

$$\mathcal{P} = \mathcal{I} - \mathcal{A}\mathcal{Q}, \qquad \mathcal{P} \in \mathbb{R}^{n\times n}, \qquad \mathcal{Q} \in \mathbb{R}^{n\times n},$$
$$\mathcal{Q} = \mathcal{Z}\mathcal{E}^{-1}\mathcal{Z}^T, \qquad \mathcal{Z} \in \mathbb{R}^{n\times k}, \qquad \mathcal{E} \in \mathbb{R}^{k\times k},$$
$$\mathcal{E} = \mathcal{Z}^T\mathcal{A}\mathcal{Z} \text{ (Tang 2008, [3])}.$$

Convergence

Deflated system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa_{eff}(\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{P}\mathcal{A})} + 1}\right)^k.$$

Deflated and preconditioned system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} + 1}\right)^k.$$

$$\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A}) \leq \kappa_{eff}(\mathcal{P}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

Deflation vectors

Recycling deflation (Clemens 2004, [4]).

$$\mathcal{Z} = [\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^{q-1}],$$

$\mathbf{x}^i$'s are solutions of the system.

Multigrid and multilevel (Tang 2009, [5]).
The matrices $\mathcal{Z}$ and $\mathcal{Z}^T$ are the restriction and prolongation matrices of multigrid methods.

Subdomain deflation (Vuik 1999,[6]).

Deflation Vectors

Model Order Reduction (MOR)

Many modern mathematical models of real-life processes pose challenges when used in numerical simulations, due to complexity and large size.

Model order reduction aims to lower the computational complexity of such problems by a reduction of the model's associated state space dimension or degrees of freedom, an approximation to the original model is computed. (Vuik 2005, [7])

- Proper Orthogonal Decomposition (POD)
- Reduced Basis Method (RBM)
- Principal Component Analysis (PCA)
- Singular Value Decomposition (SVD)

Deflation vectors

Proposal

Use solutions of the system with diverse well configurations '*snapshots*' as deflation vectors (Recycling deflation).

Use as deflation vectors the basis obtained from Proper Orthogonal Decomposition (POD).

Proper Orthogonal Decomposition (POD)

POD: find an 'optimal' basis for a given data set (Markovinović 2009 [8], Astrid 2011, [9])

- Get the snapshots

$$\mathcal{X} = [\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m].$$

- Form $\mathcal{R}$

$$\mathcal{R} := \frac{1}{m}\mathcal{X}\mathcal{X}^T \equiv \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_i\mathbf{x}_i^T.$$

- Then

$$\mathbf{\Phi} = [\phi_1, \phi_2, ...\phi_l] \in \mathbb{R}^{n\times l}$$

are the l eigenvectors corresponding to the largest eigenvalues of $\mathcal{R}$ satisfying:

$$\frac{\sum_{j=1}^{l}\lambda_j}{\sum_{j=1}^{m}\lambda_j} \leq \alpha, \qquad 0 < \alpha \leq 1.$$

Lemma 1

Let $\mathcal{A} \in \mathbb{R}^{n \times n}$ be a non-singular matrix, and $\mathbf{x}$ is a solution of:

$$\mathcal{A}\mathbf{x} = \mathbf{b}. \tag{3}$$

Let $\mathbf{x}_i, \mathbf{b}_i \in \mathbb{R}^n$, $i = 1, ..., m$, be vectors linearly independent ($l.i.$) and

$$\mathcal{A}\mathbf{x}_i = \mathbf{b}_i. \tag{4}$$

The following equivalence holds

$$\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \qquad \Leftrightarrow \qquad \mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i. \tag{5}$$

Lemma 2

If the the deflation matrix $\mathcal{Z}$ is constructed with a set of m vectors

$$\mathcal{Z} = \begin{bmatrix} \mathbf{x}_1 & ... & ... & \mathbf{x}_m \end{bmatrix}, \tag{6}$$

such that $\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i$, with $\mathbf{x}_i$ *l.i.*, then the solution of system (3) is obtained with one iteration of DCG.

Heterogeneous permeability (Neumann and Dirichlet boundary conditions).

The experiments were performed for single-phase flow, with the following characteristics:

$nx = ny = 64$ grid cells.

5 linearly independent snapshots.

System configuration								
Well pressures (bars)					Boundary conditions (bars)			
	$W1$	$W2$	$W3$	$W4$	$P(y=0)$	$P(y=Ly)$	$\frac{\partial P(x=0)}{\partial n}$	$\frac{\partial P(x=Lx)}{\partial n}$
	-5	-5	+5	+5	0	3	0	0
Snapshots								
	$W1$	$W2$	$W3$	$W4$	$P(y=0)$	$P(y=Ly)$	$\frac{\partial P(x=0)}{\partial n}$	$\frac{\partial P(x=Lx)}{\partial n}$
$\mathbf{z}_1$	-5	0	0	0	0	0	0	0
$\mathbf{z}_2$	0	-5	0	0	0	0	0	0
$\mathbf{z}_3$	0	0	-5	0	0	0	0	0
$\mathbf{z}_4$	0	0	0	-5	0	0	0	0
$\mathbf{z}_5$	0	0	0	0	0	3	0	0

Table : Table with the well configuration and boundary conditions of the system and the snapshots used for the Case 1.

Numerical experiments

Heterogeneous permeability (Neumann and Dirichlet boundary conditions).

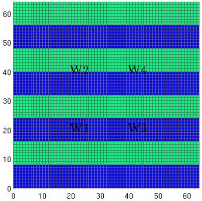

Figure : Heterogeneous permeability, 4 wells.

contrast	10^1	10^2	10^3
ICCG	75	103	110
DICCG	1	1	1

Table : Number of iterations for different contrasts between the permeability of the layers for the ICCG and DICCG methods.

Numerical experiments

Heterogeneous permeability (Neumann boundary conditions).

The experiments were performed for single-phase flow, with the following characteristics:

- $nx = ny = 64$ grid cells.
- Neumann boundary conditions.
- 15 snapshots, 4 linearly independent.
- W1 = W2 = W3 = W4 = -1 bars, W5 = +4 bars.

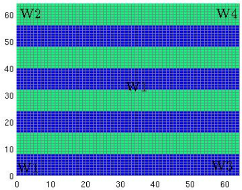

Figure : Heterogeneous permeability layers.

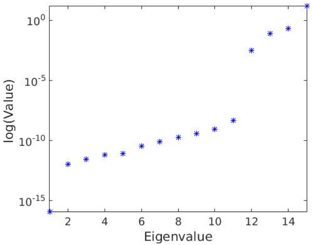

Figure : Eigenvalues of the data snapshot correlation matrix.

contrast	10^1	10^2	10^3
ICCG	90	115	131
$DICCG_4$	1	1	1
$DICCG_{15}$	200*	200*	200*
$DICCG_{POD_4}$	1	1	1

Table : Number of iterations.

SPE 10 model

60x220x85 grid cells.

Neumann boundary conditions.

15 snapshots, 4 linearly independent.

W1 = W2 = W3 = W4 = -1 bars, W5 = +4 bars.

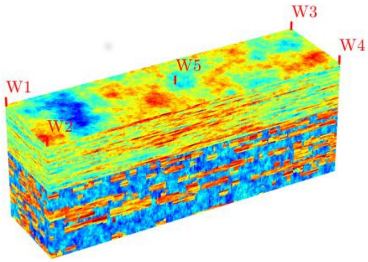

Figure : SPE 10 benchmark, permeability field.

Method	Iterations
ICCG	1011
$DICCG_{15}$	2000*
$DICCG_4$	2
$DICCG_{POD_4}$	2

Table : Number of iterations for ICCG and DICCG methods.

Compressible heterogeneous layered problem

35x35 grid cells.

Neumann boundary conditions.

W1 = W2 = W3 = W4 = 100 bars, W5 = 600 bars.

Initial pressure 200 bars.

Contrast between permeability layers of 10^1, 10^2 and 10^3.

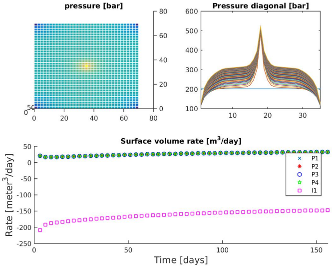

Figure : Solution, contrast between permeability layers of 10^1.

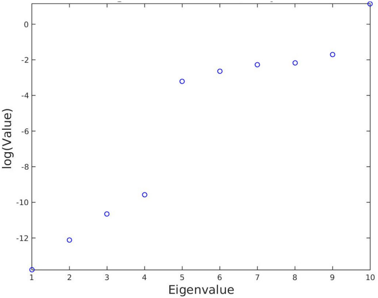

Figure : Eigenvalues of the data snapshot correlation matrix, contrast between permeability layers of 10^1.

Numerical experiments

1st NR Iteration						
$\frac{\sigma_2}{\sigma_1}$	Total ICCG(only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG(only)
10^1	780	DICCG_{10}	140	42	182	23
	780	DICCG_{POD_6}	140	84	224	29
10^2	624	DICCG_{10}	100	42	142	23
	624	DICCG_{POD_7}	100	42	142	23
10^3	364	DICCG_{10}	20	42	62	17
	364	DICCG_{POD_7}	20	42	62	17

Table : Comparison between the ICCC and DICCG methods of the average number of linear iterations for the first NR iteration for various contrast between permeability layers.

Compressible SPE 10 problem

60x220x85 grid cells.

Neumann boundary conditions.

W1 = W2 = W3 = W4 = 100 bars, W5 = 600 bars.

Initial pressure 200 bars.

Contrast in permeability of 3x10^7.

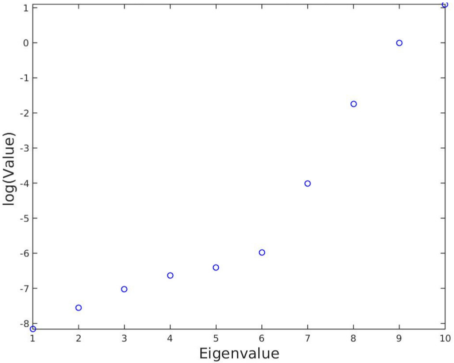

Figure : Eigenvalues of the data snapshot correlation matrix.

Numerical experiments

1st NR Iteration					
Total ICCG(only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG(only)
10173	$DICCG_{10}$	1770	1134	2904	28
10173	$DICCG_{POD_4}$	1770	1554	3324	32

Table : Total number of linear iterations for the first NR iteration, full SPE 10 benchmark.

2nd NR Iteration					
Total ICCG(only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG(only)
10231	$DICCG_{10}$	1830	200	2030	20
10231	$DICCG_{POD_4}$	1830	200	2030	20

Table : Total number of linear iterations for the second NR iteration, full SPE 10 benchmark.

Conclusions

- Solution is reached in few (1 or 2) iterations for the DICCG method in the incompressible case.
- A good choice of snapshots takes into account the boundary conditions of the problem.
- The number of iterations of the DICCG method does not depend on the contrast between the coefficients (Heterogeneous permeability example).
- The number of iterations of the ICCG method is reduced up to 80% with the DICCG method in the compressible case.
- Only a limited number of POD basis vectors is necessary to obtain a good speed-up. (for more info see [10, 11])

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