Efficient preconditioners for the Helmholtz equations

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Contents

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1. Introduction

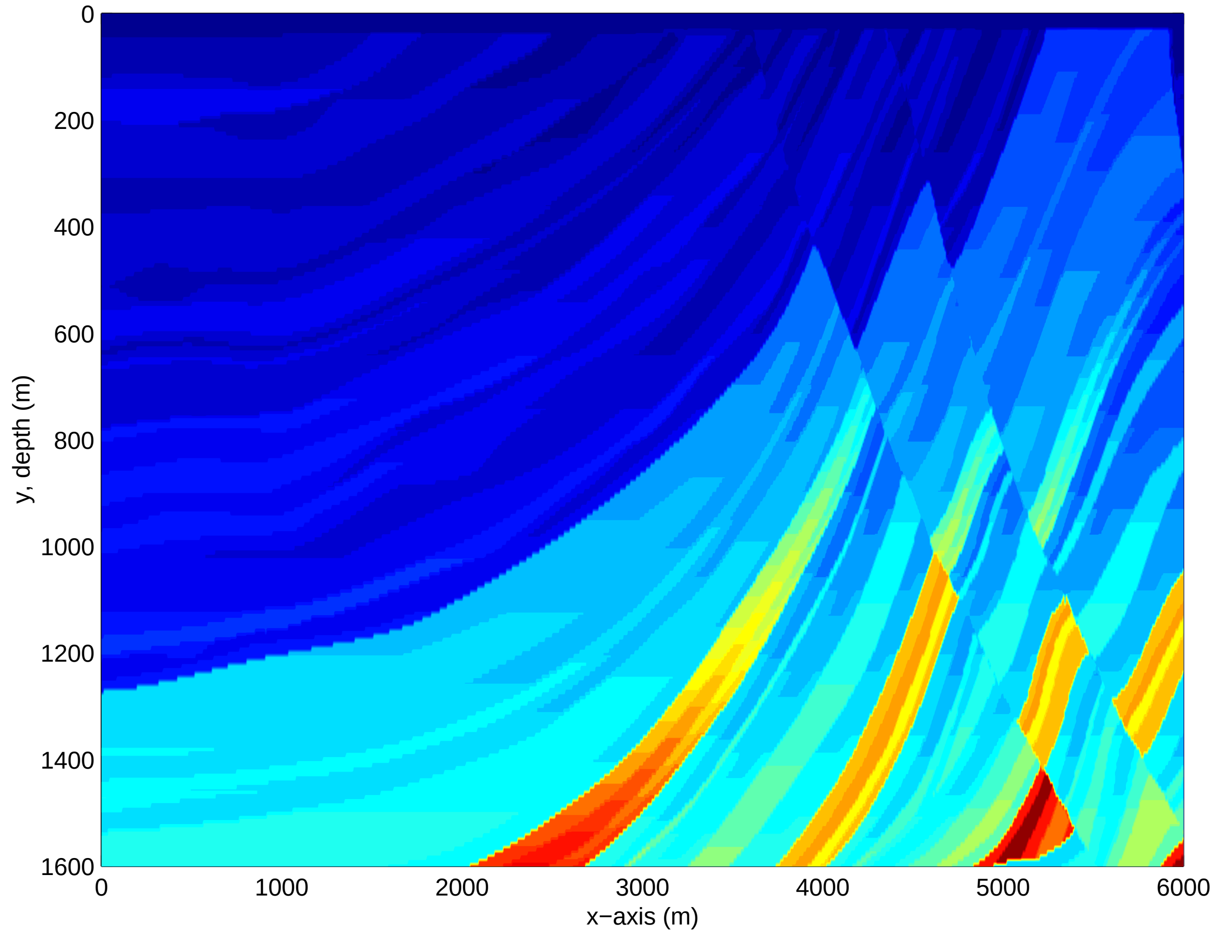
The Helmholtz problem is defined as follows

$$-\partial_{xx}u - \partial_{yy}u - (1-\alpha\mathrm{i})k^2(x,y)u = f, \quad \text{in} \quad \Omega,$$
$$\text{Boundary condition} \quad \text{on} \quad \Gamma = \partial\Omega,$$

where:

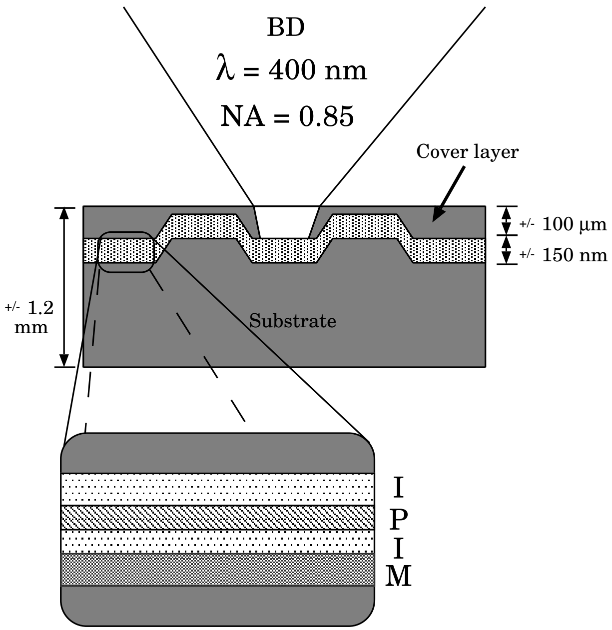
- $k(x,y)$ is the wavenumber
- for ”solid” boundaries: Dirichlet/Neumann
- for ”fictitious” boundaries: Sommerfeld $\frac{du}{dn} - \mathrm{i}ku = 0$

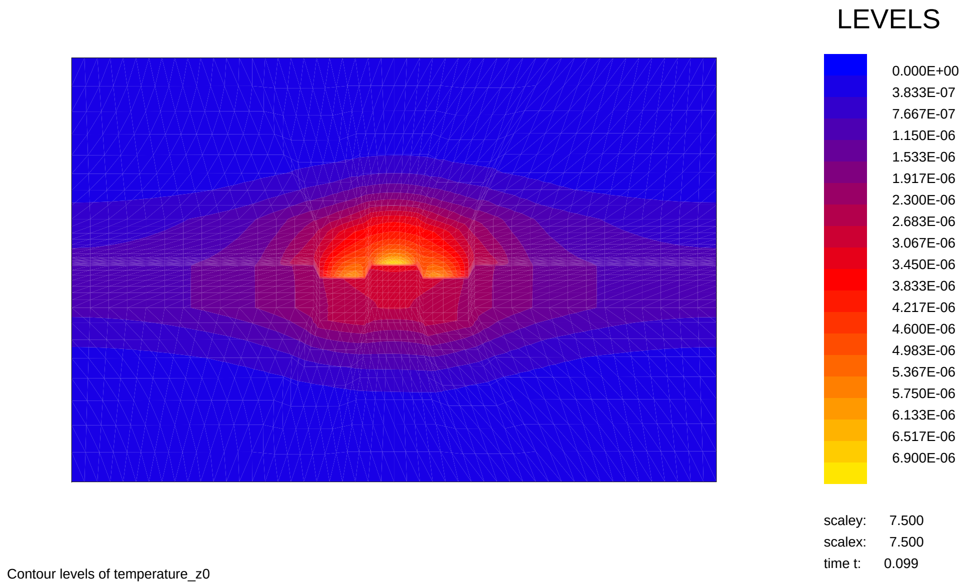
Application: geophysical survey, hard Marmousi Model



Application: optical storage

Model for Blu-Ray disk





Discretization

In general: Finite Difference/Finite Element Methods.

Particular to the present case: 5-point Finite Difference stencil, $\mathcal{O}(h^2)$.

Linear system

$$Ax = b, \;\; A \in \mathbb{C}^{N \times N}, \;\; b, x \in \mathbb{C}^N,$$

Discretization

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Linear system

$$Ax = b, \;\; A \in \mathbb{C}^{N \times N}, \;\; b, x \in \mathbb{C}^N,$$

A is a sparse, highly indefinite matrix for practical values of k.
Special property $A = A^T$.

For high resolution a very fine grid is required: $30 - 60$ gridpoints per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A$ is extremely large!

Characteristic properties of the problem

- $A \in \mathbb{C}^{N \times N}$ is sparse
- wavenumber k and grid size N are very large
- wavenumber k varies discontinuously
- real parts of the eigenvalues of A are positive and negative

2. Survey of solution methods

Special Krylov methods

- COCG van der Vorst and Melissen, 1990
- QMR Freund and Nachtigal, 1991

General purpose Krylov methods

- CGNR Paige and Saunders, 1975
- Short recurrences

  CGS Sonneveld, 1989

  Bi-CGSTAB van der Vorst, 1992

- Minimal residual

  GMRES Saad and Schultz, 1986

  GCR Eisenstat, Elman and Schultz, 1983

  GMRESR van der Vorst and Vuik, 1994

Survey of preconditioners

ILU	Meijerink and van der Vorst, 1977
ILU(tol)	Saad, 2003
SPAI	Grote and Huckle, 1997
Multigrid	Lahaye, 2001
	Elman, Ernst and O' Leary, 2001
AILU	Gander and Nataf, 2001
	analytic parabolic factorization
ILU-SV	Plessix and Mulder, 2003
	separation of variables

3. Shifted Laplacian preconditioners with an SPD real part

Operator based preconditioner M is based on discrete version of

$$-\partial_{xx}u - \partial_{yy}u - (\beta_1 - \beta_2 \mathrm{i})k^2(x,y)u = f, \quad \text{in}\Omega.$$

Matrix M^{-1} is approximated by an inner iteration process.

To obtain a fast inner iteration we assume that $\beta_1 \leq 0$. This implies that real part of M is SPD.

$\beta_1 = 0$	$\beta_2 = 0$	Laplacian	Bayliss and Turkel, 1983
$\beta_1 = -1$	$\beta_2 = 0$	Definite Helmholtz	Laird, 2000
$\beta_1 = 0$	$\beta_2 = 1$	Shifted Laplacian	Erlangga, Vuik and Oosterlee, 2003

Eigenvalues of the various preconditioned matrices

Denote $Q = (M^{-1}A)^*(M^{-1}A)$ and the eigenvalues of the Laplacian part B as

$$0 < \mu_1 \leq \mu_2 \cdots \leq \mu_n.$$

Bayliss and Turkel	$\lambda_j(Q_0) = \left(1 - \frac{k^2}{\mu_j}\right)^2,$
Laird	$\lambda_j(Q_1) = \left(1 - \frac{2k^2}{\mu_j + k^2}\right)^2,$
Complex	$\lambda_j(Q_i) = 1 - \frac{2\mu_j k^2}{\mu_j^2 + k^4}.$

Comparison of the eigenvalues for $k^2 < \mu_1$

After some analysis, the following inequalities are derived:

$$\lambda_{\min}(Q_0) > \lambda_{\min}(Q_1),$$

$$\lambda_{\min}(Q_0) > \lambda_{\min}(Q_i),$$

and

$$\lim_{\mu_n \to \infty} \lambda_{\max}(Q_0) = \lim_{\mu_n \to \infty} \lambda_{\max}(Q_1) = \lim_{\mu_n \to \infty} \lambda_{\max}(Q_i) = 1$$

Conclusion

For low k, M_0 performs better than M_1 and M_i.

Eigenvalues for Bayliss and Turkel preconditioner for $\mu_1 < k^2 < \mu_n$

The smallest eigenvalue

$$\lambda_{\min}(Q_0) = \frac{\epsilon^2}{k^4}$$

and for small k

$$\lim_{\mu_n \to \infty} \lambda_n(Q_0) = 1 \text{ and } \lim_{\mu_1 \to 0} \lambda_1(Q_0) = \infty$$

for large k

$$\lim_{k \to \infty} \lambda_{\max}(Q_0) = \infty.$$

Remark

There is a possible unboundedness for large k.

Eigenvalues for Laird preconditioner for $\mu_1 < k^2 < \mu_n$

The smallest eigenvalue

$$\lambda_{\min}(Q_1) = \frac{\epsilon^2}{4k^4}$$

and

$$\lim_{\mu_n \to \infty} \lambda_n(Q_1) = 1, \lim_{\mu_1 \to 0} \lambda_1(Q_1) = 1, \text{ and } \lim_{k \to \infty} \lambda_{\max}(Q_1) = 1.$$

Remark

The eigenvalues are always bounded above by one, but some small eigenvalues lie very close to the origin $\rightarrow$ the cause of slow convergence!

Eigenvalues for Complex preconditioner for $\mu_1 < k^2 < \mu_n$

The smallest eigenvalue

$$\lambda_{\min}(Q_i) = \frac{\epsilon^2}{2k^4}$$

$$\lim_{\mu_n \to \infty} \lambda_n(Q_i) = 1, \lim_{\mu_1 \to 0} \lambda_1(Q_i) = 1, \text{ and } \lim_{k \to \infty} \lambda_{\max}(Q_i) = 1.$$

Remark

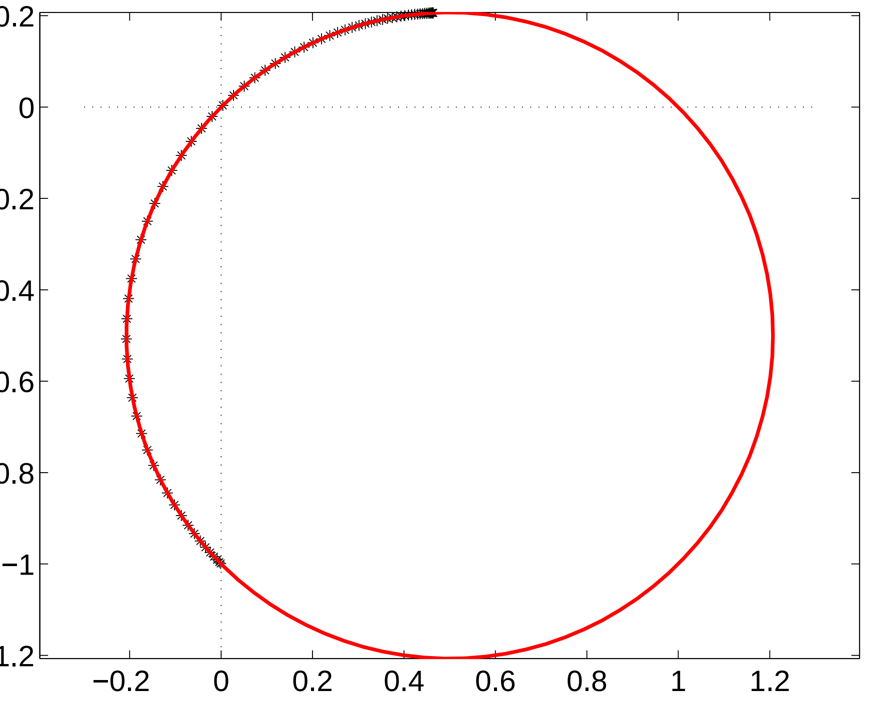
The eigenvalues are always bounded above by one. Some small eigenvalues lie very close to the origin **BUT** are still farther away as compared to those of M_1.

Conclusion

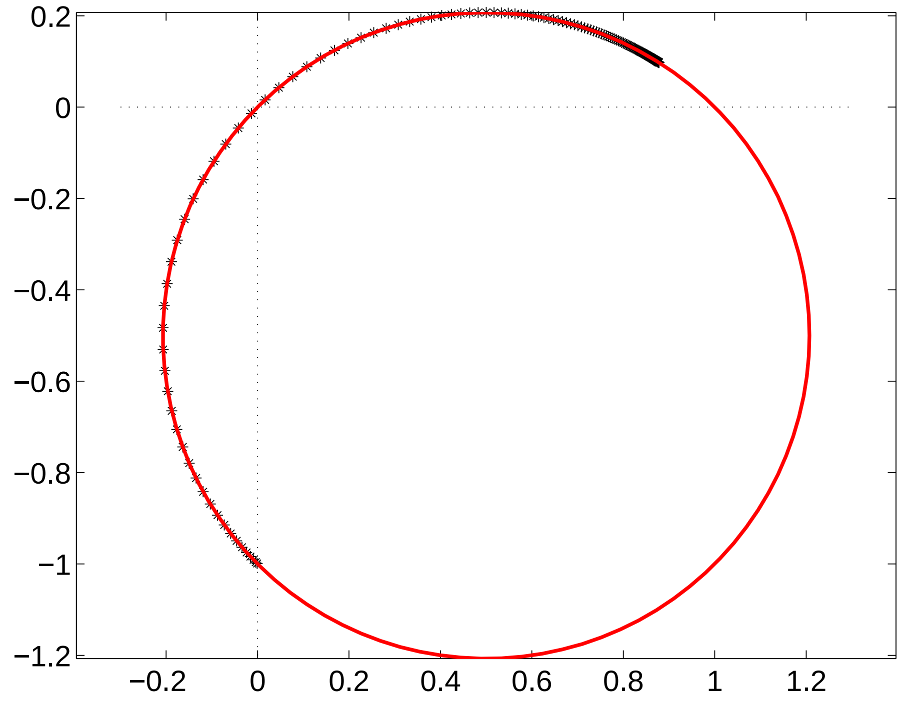
For large k, M_i may be better than M_0 and M_1.

Eigenvalues for Complex preconditioner $k = 100$

75 grid points



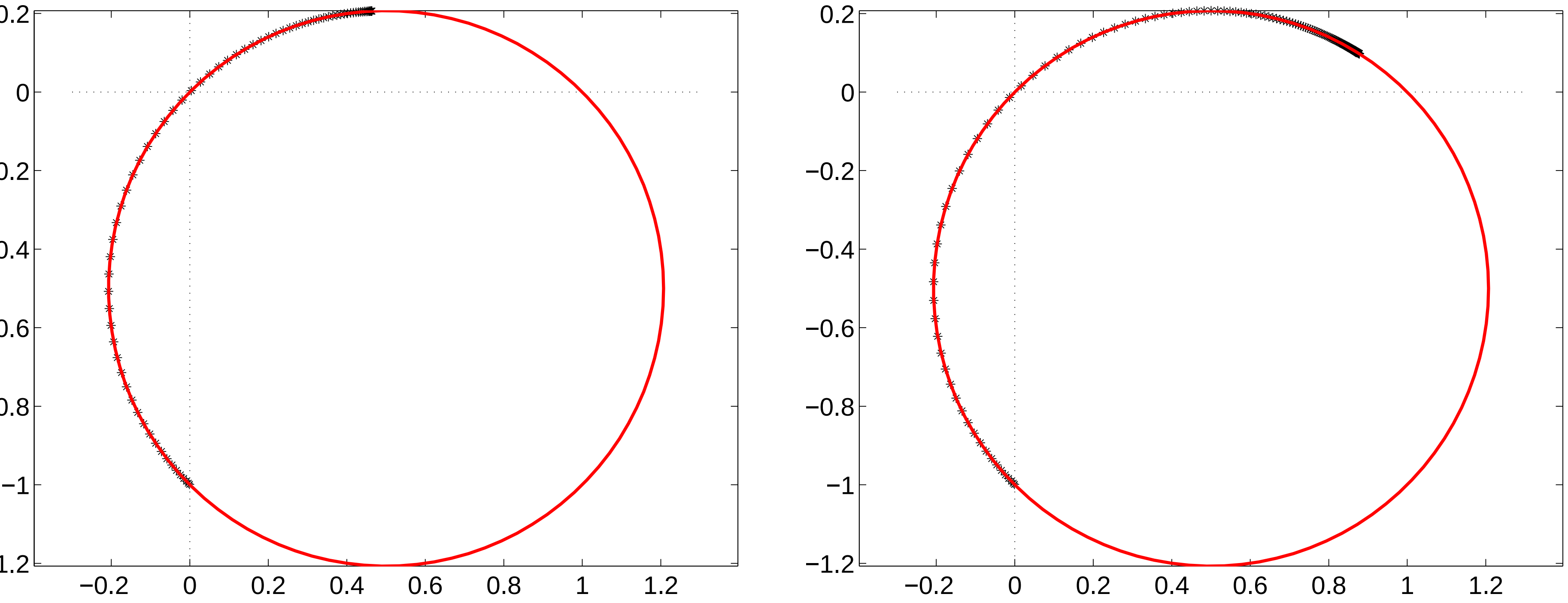
150 grid points



Eigenvalues for Complex preconditioner $k = 100$

Spectrum is independent of the grid size

75 grid points   150 grid points



4. General Shifted Laplacian preconditioners

Assumption $\beta_1 \leq 0$.

Possible solvers for solution of $Mz = r$:

- ILU approximation of M
- inner iteration with ILU as preconditioner
- Multigrid

Multigrid components
- geometric multigrid
- Gauss-Seidel with red-black ordering
- matrix dependent interpolation, full weighting restriction
- Galerkin coarse grid approximation

No restriction on β_1

For the outer loop $\beta_1 = 1$ and $\beta_2 = 0$ is optimal. Convergence in 1 iteration. **But**, the innerloop does not converge with multigrid (original problem).

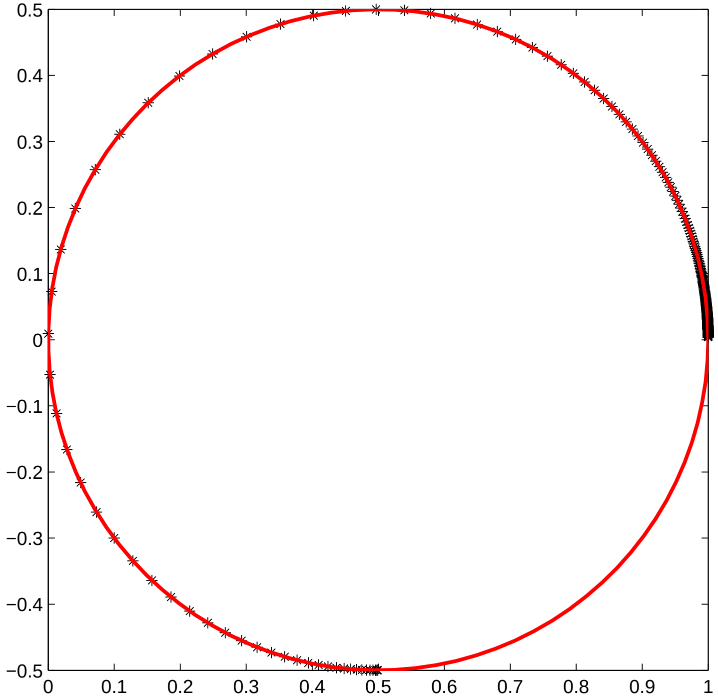
However, it appears that multigrid works well for $\beta_1 = 1$ and $\beta_2 = 1$ and

the convergence of the outer loop is much faster.

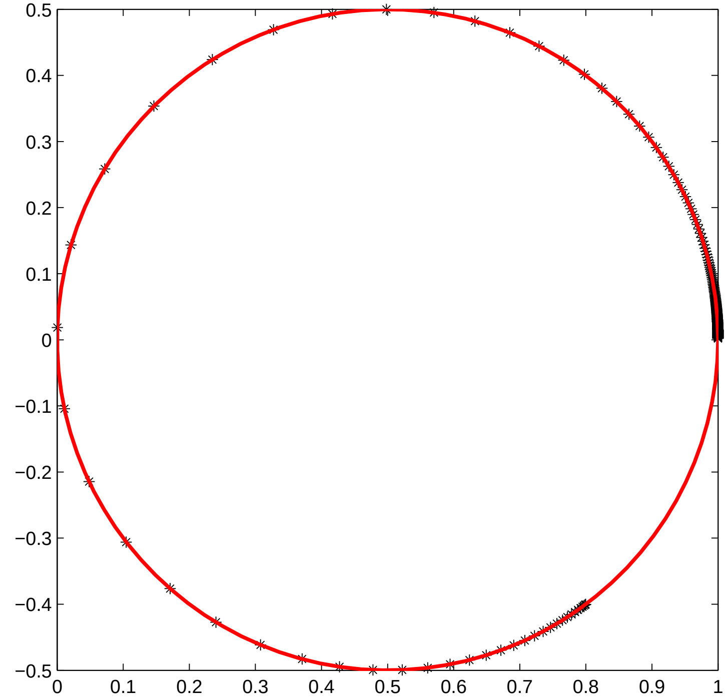
Eigenvalues for Complex preconditioner $k = 100$ *and* $\beta_1 = 1$

Spectrum is independent of the grid size

$\beta_2 = 1$



$\beta_2 = 0.5$



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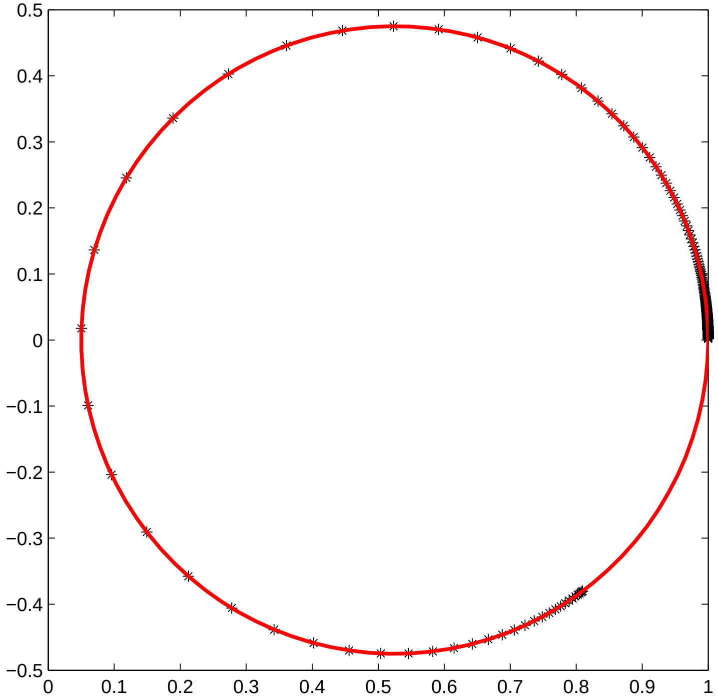
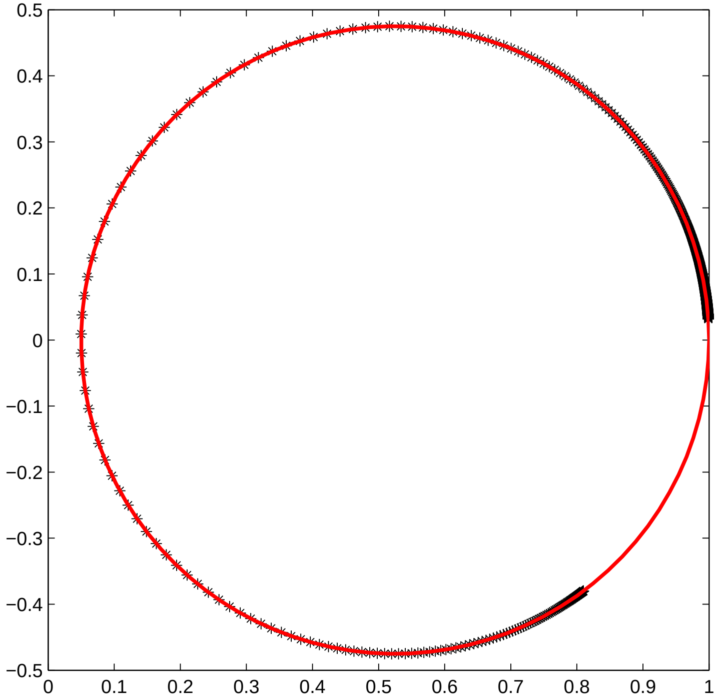
Eigenvalues for $\alpha = 0.025$ (phys. damping) and $\beta_1 = 1$ and $\beta_2 = 0.5$

Spectrum is independent of the grid size and the choice of k.

$k = 100$



$k = 400$



Multigrid components

- geometric multigrid
- ω-JAC smoother
- bilinear interpolation, restriction operator full weighting
- Galerkin coarse grid approximation
- F(1,1)-cycle
- M^{-1} is approximated by *one* multigrid iteration

5. Numerical results

	CGNR			Bi-CGSTAB		
k_{ref}	M_0	M_1	M_i	M_0	M_1	M_i
2	12	12	10	6	7	5
5	39	31	23	17	15	13
10	189	88	66	150	56	22
15	647	175	126	685	113	40
20	$>$1000	268	194	$>$1000	177	60
30	$>$1000	502	361	$>$1000	344	105

Numerical results for a wedge problem

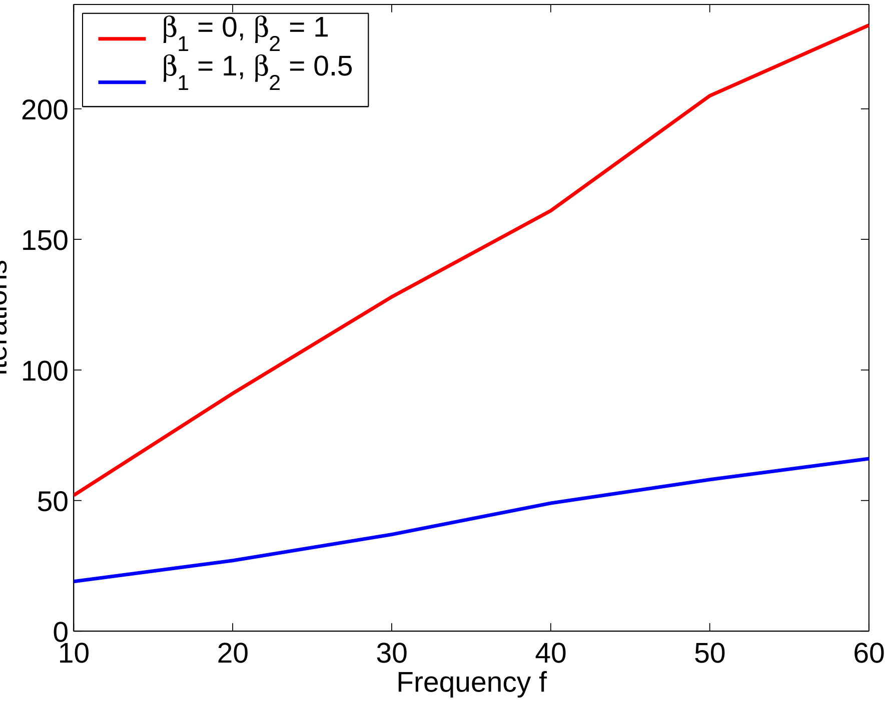
k_2	10	20	40	50	100
grid	32^2	64^2	128^2	192^2	384^2
No-Prec	201(0.56)	1028(12)	5170(316)	–	–
ILU(A,0)	55(0.36)	348(9)	1484(131)	2344(498)	–
ILU(A,1)	26(0.14)	126(4)	577(62)	894(207)	–
ILU(M,0)	57(0.29)	213(8)	1289(122)	2072(451)	–
ILU(M,1)	28(0.28)	116(4)	443(48)	763(191)	2021(1875)
MG(V(1,1))	13(0.21)	38(3)	94(28)	115(82)	252(850)

Numerical results for Marmousi

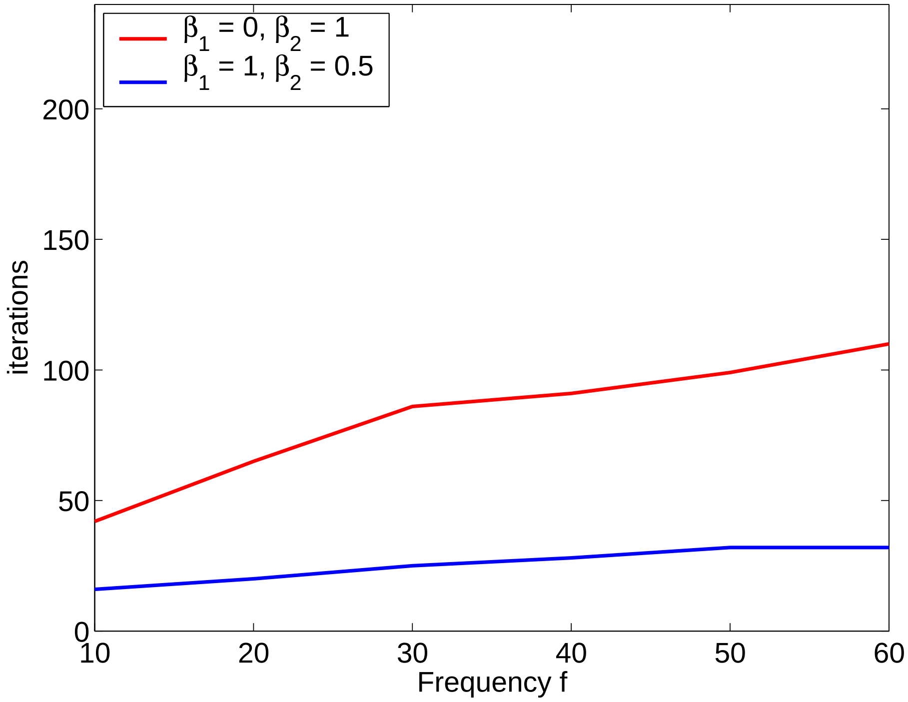
f (Hz)	1	10	20
grid	751×201	751×201	1501×401
No-Prec	$>$4000($>$2000)	–	–
ILU(A,0)	2336(1850)	break	3981(12692)
ILU(M,0)	3124(2487)	1513(1318)	3376(10710)
MG(V(1,0))	18(50)	175(401)	320(4440)
MG(V(1,1))	14(50)	169(499)	318(5344)
MG(F(1,0))	15(48)	160(550)	312(4645)
SV [APNUM 44(2003)]	3(–)	114(–)	648(–)

Numerical results for Marmousi

no damping ($\alpha = 0$)



damping ($\alpha = 0.05$)



6. Conclusions

- The shifted Laplacian operator leads to robust preconditioners for the Helmholtz equation.
- For small k the Bayliss & Turkel preconditioner is optimal.
- For large k the complex shifted Laplacian preconditioner is optimal.
- The proposed preconditioner (shifted Laplacian + multigrid) is independent of the grid size and linearly dependent of k.
- With physical damping the proposed preconditioner is also independent of k.
- The choice $(\beta_1 = 1, \beta_2 = 0.5)$ is 9-18 times faster than an ILU preconditioner for a realistic test problem (Marmousi).

Further information/research

- http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_helmholtz.html

- Y.A. Erlangga and C. Vuik and C.W. Oosterlee
  On a class of preconditioners for solving the Helmholtz equation
  Appl. Num. Math., 50, pp. 409-425, 2004

- Current research
  - 3 dimensional problem
  - matrix free implementation
  - parallel implementation
  - finite element discretization