# ”Shifted Laplace” preconditioners for the Helmholtz equations


C. Vuik, Y.A. Erlangga, and C.W. Oosterlee

`c.vuik@math.tudelft.nl`

`http://ta.twi.tudelft.nl/users/vuik/`

Delft University of Technology



ENUMATH 2003

The European Conference on Numerical Mathematics and Advanced Applications

August 18-22, 2003, Prague, Czech Republic


TUDelft

# *Contents*

1. Introduction
2. Survey of solution methods
3. Survey of preconditioners
4. Properties of the ”Shifted Laplace” preconditioners
5. Numerical experiments
6. Conclusions

Wim Mulder, René Edouard Plessix, Paul Urbach

Financially supported by the Dutch Ministry of Economic Affairs: project BTS01044

TU Delft

# 1. Introduction

The Helmholtz problem is defined as follows

$$\Delta u + k^2 u = f, \quad \text{in} \quad \Omega,$$
$$\text{Boundary condition} \quad \text{on} \quad \Gamma = \partial\Omega,$$

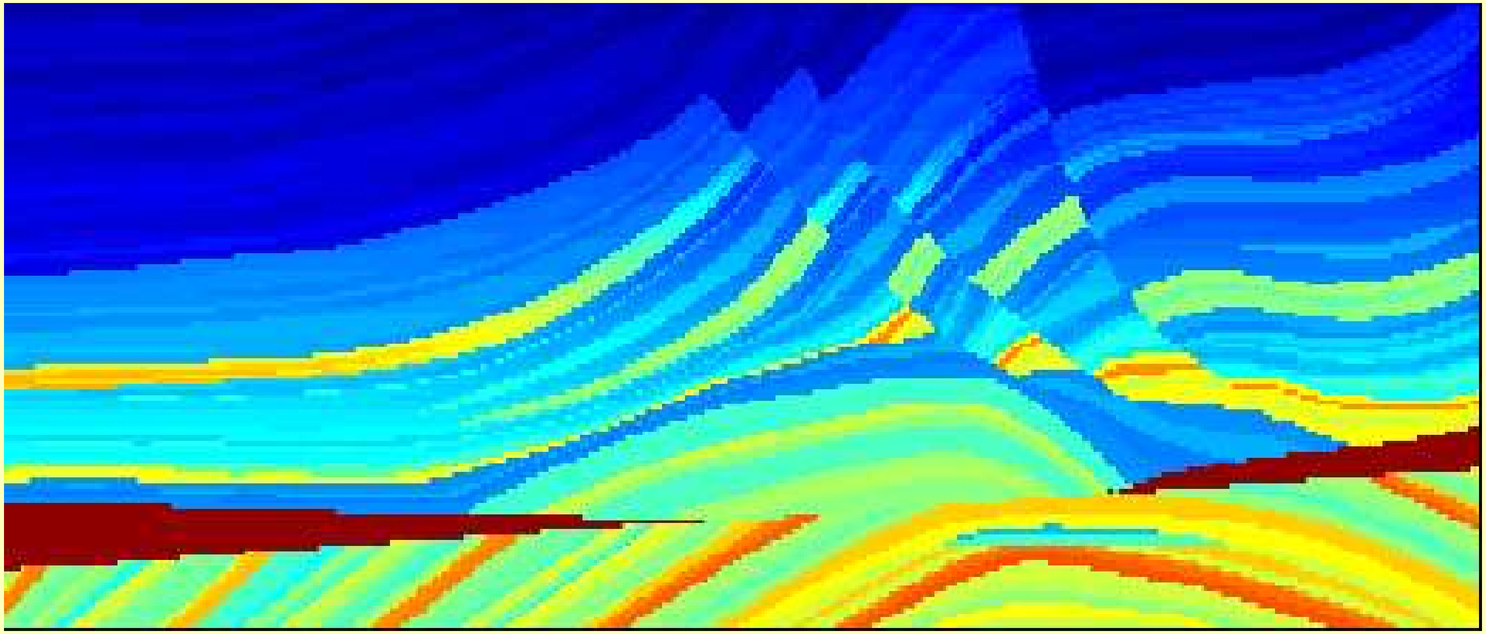
where:

- $k = k(x, y, z)$ is the wavenumber
- for ”solid” boundaries: Dirichlet/Neumann
- for ”fictitious” boundaries: Sommerfeld $\frac{du}{dn} - \mathrm{i}ku = 0$

TUDelft

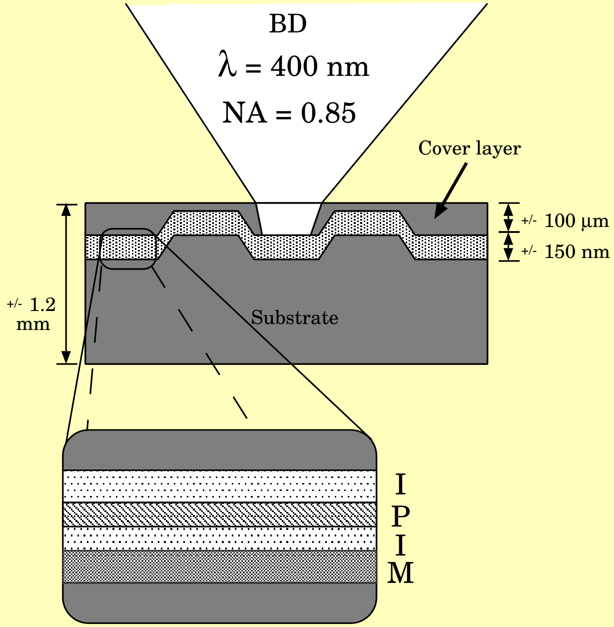
# *Application: geophysical survey*

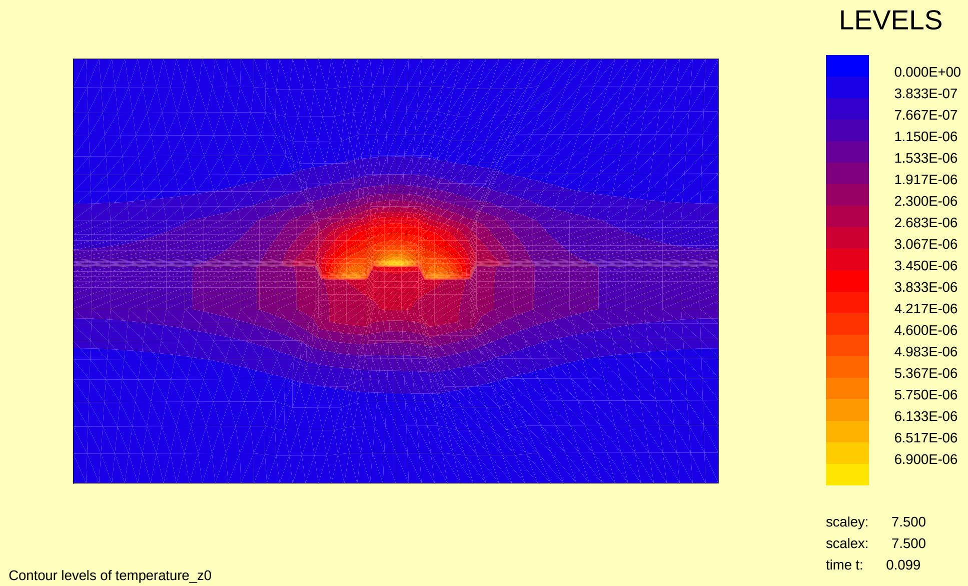
hard Marmousi Model



# *Application: optical storage*

## Model for Blu-Ray disk




TU Delft

# *Discretization*

In general: Finite Difference/Finite Element Methods.

Particular to the present case: 5-point Finite Difference stencil, $\mathcal{O}(h^2)$.

Linear system

$$Ax = b, \;\; A \in \mathbb{C}^{N \times N}, \;\; b, x \in \mathbb{C}^N,$$

TU Delft

# *Discretization*

In general: Finite Difference/Finite Element Methods.

Particular to the present case: 5-point Finite Difference stencil, $\mathcal{O}(h^2)$.

Linear system

$$Ax = b, \;\; A \in \mathbb{C}^{N\times N}, \;\; b, x \in \mathbb{C}^N,$$

$A$ is a sparse, highly indefinite matrix for practical values of $k$.
Special property $A = A^T$.

For high resolution a very fine grid is required: $30 - 60$ gridpoints per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A$ is extremely large!

TUDelft

# 2. Survey of solution methods

Special Krylov methods

- COCG van der Vorst and Melissen, 1990
- QMR Freund and Nachtigal, 1991

TU Delft

# *2. Survey of solution methods*

Special Krylov methods

- COCG van der Vorst and Melissen, 1990
- QMR Freund and Nachtigal, 1991

General purpose Krylov methods

- CGNR Paige and Saunders, 1975
- Short recurrences
  CGS Sonneveld, 1989
  Bi-CGSTAB van der Vorst, 1992
- Minimal residual
  GMRES Saad and Schultz, 1986
  GCR Eisenstat, Elman and Schultz, 1983
  GMRESR van der Vorst and Vuik, 1994

TU Delft

# 3. Survey of preconditioners

Equivalent linear system $M_1^{-1}AM_2^{-1}\tilde{x} = \tilde{b}$, where $M = M_1 \cdot M_2$ is the preconditioning matrix and

$$\tilde{x} = M_2 x, \quad \tilde{b} = M_1 b.$$

Requirements for a preconditioner

- better spectral properties of $M^{-1}A$
- cheap to perform $M^{-1}r$.

Spectrum of $A$ is $\{\mu_i - k^2\}$, with $k$ is constant and $\mu_i$ are the eigenvalues of the Laplace operator. Note $\mu_1 - k^2$ may be negative.

TUDelft

# *Survey of preconditioners*

| | |
|---|---|
| ILU | Meijerink and van der Vorst, 1977 |
| ILU(tol) | Saad, 2003 |
| | |
| SPAI | Grote and Huckle, 1997 |
| Multigrid | Lahaye, 2001 |
| | Elman, Ernst and O' Leary, 2001 |

TU Delft

# *Survey of preconditioners*

| | |
|---|---|
| ILU | Meijerink and van der Vorst, 1977 |
| ILU(tol) | Saad, 2003 |
| SPAI | Grote and Huckle, 1997 |
| Multigrid | Lahaye, 2001 |
| | Elman, Ernst and O' Leary, 2001 |
| AILU | Gander and Nataf, 2001 |
| | analytic parabolic factorization |
| ILU-SV | Plessix and Mulder, 2003 |
| | separation of variables |

## *Survey of preconditioners*

| | |
|---|---|
| Laplace operator | Bayliss and Turkel, 1983 |
| Definite Helmholtz | Laird, 2000 |
| Shifted Laplace | Y.A. Erlangga, C. Vuik and C.W.Oosterlee, 2003 |

TU Delft

# *Survey of preconditioners*

| | |
|---|---|
| Laplace operator | Bayliss and Turkel, 1983 |
| Definite Helmholtz | Laird, 2000 |
| Shifted Laplace | Y.A. Erlangga, C. Vuik and C.W.Oosterlee, 2003 |

Shifted Laplace preconditioner

$$M \equiv \Delta - (\alpha + \mathrm{i}\beta)k^2, \ \ \alpha, \beta \in \mathbb{R}, \ \ \text{and} \ \ \alpha \geq 0.$$

Condition $\alpha \geq 0$ is used to ensure that $M$ is a (semi) definite operator.

$\rightarrow \alpha, \beta = 0$ : Bayliss and Turkel

$\rightarrow \alpha = 1, \beta = 0$ : Laird

TU Delft

## 4. Properties of the "Shifted Laplace" preconditioners

### Motivation: continuous problem

Consider 1D Helmholtz problem with Dirichlet boundary conditions.
Continuous generalized eigenvalue problem:

$$\left(\frac{d^2}{dx^2} + k^2\right)\phi_v = \lambda\left(\frac{d^2}{dx^2} - (\alpha + \mathrm{i}\beta)k^2\right)\phi_v$$

Eigenvalues:

$$\lambda_n = \frac{k_n^2 - k^2}{k_n^2 + (\alpha + \mathrm{i}\beta)k^2}, \quad k_n = n\pi, \quad n \in \mathbb{N}\backslash\{0\},$$

or, in modulus,

$$|\lambda_n|^2 = \frac{(k_n - k^2)^2}{(k_n^2 + \alpha k^2)^2 + \beta^2 k^4}.$$

TUDelft

# *Spectral properties ($\alpha^2 + \beta^2 \neq 0$)*

**Maximum $|\lambda|$**

For finite $k$ and $k_n \to \infty$: $|\lambda_n|^2 = 1$, and for $k \to \infty$: $|\lambda_1|^2 = \frac{1}{\alpha^2+\beta^2}$, so

$$|\lambda_{\max}|^2 = \max\left(\frac{1}{\alpha^2+\beta^2}, 1\right).$$

**Minimum $|\lambda|$**

Assume $|\lambda_{\min}| \approx 0$. This implies $k_j \approx k$, or $k_j = k + \epsilon$.
Substitution leads to:

$$|\lambda_{\min}|^2 = \frac{4}{(1+\alpha)^2+\beta^2}\left(\frac{\epsilon}{k}\right)^2.$$

TUDelft

# *Condition number*

For $\alpha = 0$ and $\beta = 0$, condition number $\kappa^2 = \frac{k^6}{4\pi^4\epsilon^2}$.

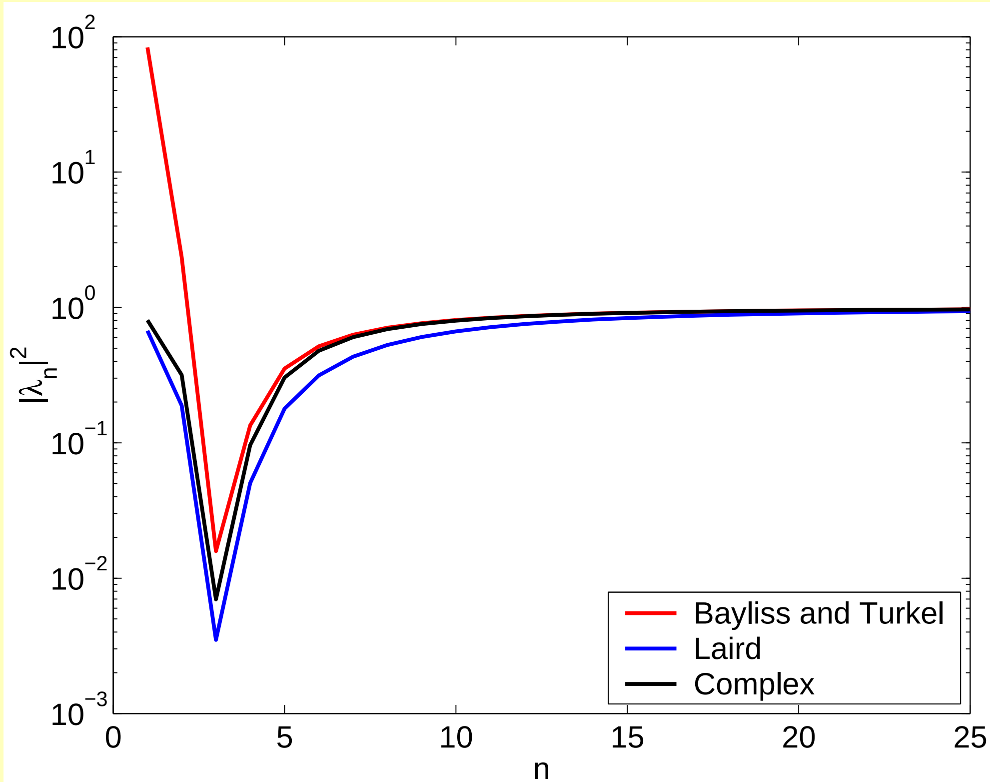
For other values of $\alpha$ and $\beta$ we have

$$\kappa^2 = \begin{cases} \frac{1}{4}\left(1 + \frac{1+2\alpha}{\alpha^2+\beta^2}\right)(k/\epsilon)^2, & \alpha^2 + \beta^2 \leq 1, \\ \frac{1}{4}((1+\alpha)^2 + \beta^2)(k/\epsilon)^2, & \alpha^2 + \beta^2 \geq 1. \end{cases}$$

By inspection

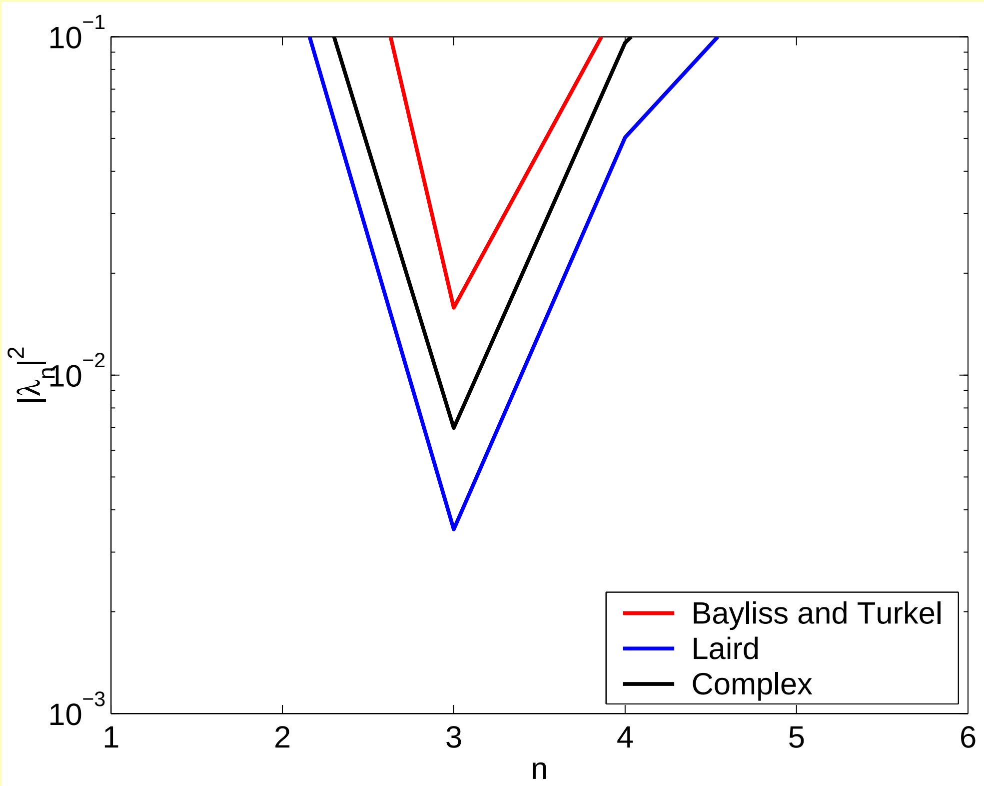
- $\kappa^2$ is minimal on the circle $\alpha^2 + \beta^2 = 1$
- with $\alpha \geq 0$, $\kappa$ is minimal for $\alpha = 0, \beta = 1$

# *Illustration: spectrum of the generalized eigenvalue problem*




TUDelft

# *Illustration: spectrum of the generalized eigenvalue problem*




Generalization to 3D problems is easy and gives the same results.

# *Spectral properties of the discretized operator*

Consider

$$M^{-1}Ax = M^{-1}b,$$

$M$ is the discretized Shifted-Laplace operator.

Introduce the splitting $A = B + k^2 I$, $B$ is the Laplace component of $A$.

Generalized eigenvalue problem:

$$\left(B + k^2 I\right) p_v = \lambda_v \left(B - (\alpha + \mathrm{i}\beta)k^2 I\right) p_v.$$

## *Spectral properties of the discretized operator*

Eigenvalues can have both positive and negative real part.

$\rightarrow$ indefinite.
$\rightarrow$ convergence is difficult to estimate

The normal equations formulation is used to estimate the convergence

$$(M^{-1}A)^*(M^{-1}A)x = (M^{-1}A)^*b$$

We consider three particular options:

| | |
|---|---|
| $\alpha = 0, \beta = 0 : M_0$ | Bayliss and Turkel |
| $\alpha = 1, \beta = 0 : M_1$ | Laird |
| $\alpha = 0, \beta = 1 : M_i$ | Complex |

TUDelft

# *Eigenvalues of the various preconditioned matrices*

Denote $Q = (M^{-1}A)^*(M^{-1}A)$ and the eigenvalues of $B$ as

$$0 < \mu_1 \leq \mu_2 \cdots \leq \mu_n.$$

| | |
|---|---|
| Bayliss and Turkel | $\lambda_j(Q_0) = \left(1 - \frac{k^2}{\mu_j}\right)^2,$ |
| Laird | $\lambda_j(Q_1) = \left(1 - \frac{2k^2}{\mu_j + k^2}\right)^2,$ |
| Complex | $\lambda_j(Q_i) = 1 - \frac{2\mu_j k^2}{\mu_j^2 + k^4}.$ |

TUDelft

## *Comparison of the eigenvalues for $k^2 < \mu_1$*

After some analysis, the following inequalities are derived:

$$\lambda_{\min}(Q_0) > \lambda_{\min}(Q_1),$$

$$\lambda_{\min}(Q_0) > \lambda_{\min}(Q_i),$$

and

$$\lim_{\mu_n \to \infty} \lambda_{\max}(Q_0) = \lim_{\mu_n \to \infty} \lambda_{\max}(Q_1) = \lim_{\mu_n \to \infty} \lambda_{\max}(Q_i) = 1$$

Conclusion

For low $k$, $M_0$ performs better than $M_1$ and $M_i$.

TUDelft

## *Eigenvalues for Bayliss and Turkel preconditioner for* $\mu_1 < k^2 < \mu_n$

The smallest eigenvalue

$$\lambda_{\min}(Q_0) = \frac{\epsilon^2}{k^4}$$

and for small $k$

$$\lim_{\mu_n \to \infty} \lambda_n(Q_0) = 1 \text{ and } \lim_{\mu_1 \to 0} \lambda_1(Q_0) = \infty$$

for large $k$

$$\lim_{k \to \infty} \lambda_{\max}(Q_0) = \infty.$$

Remark

There is a possible unboundedness for large $k$.

TUDelft

## *Eigenvalues for Laird preconditioner for* $\mu_1 < k^2 < \mu_n$

The smallest eigenvalue

$$\lambda_{\min}(Q_1) = \frac{\epsilon^2}{4k^4}$$

and

$$\lim_{\mu_n \to \infty} \lambda_n(Q_1) = 1, \lim_{\mu_1 \to 0} \lambda_1(Q_1) = 1, \text{ and } \lim_{k \to \infty} \lambda_{\max}(Q_1) = 1.$$

**Remark**

The eigenvalues are always bounded above by one, but some small eigenvalues lie very close to the origin $\rightarrow$ the cause of slow convergence!

## *Eigenvalues for Complex preconditioner for* $\mu_1 < k^2 < \mu_n$

The smallest eigenvalue

$$\lambda_{\min}(Q_i) = \frac{\epsilon^2}{2k^4}$$

$$\lim_{\mu_n \to \infty} \lambda_n(Q_i) = 1, \ \lim_{\mu_1 \to 0} \lambda_1(Q_i) = 1, \text{ and } \lim_{k \to \infty} \lambda_{\max}(Q_i) = 1.$$

Remark

The eigenvalues are always bounded above by one. Some small eigenvalues lie very close to the origin **BUT** are still farther away as compared to those of $M_1$.

Conclusion

For large $k$, $M_i$ may be better than $M_0$ and $M_1$.

TUDelft

# 5. Numerical experiments

Problem 1: Example with constant $k$ in $\Omega$

Iterative solver: Bi-CGSTAB

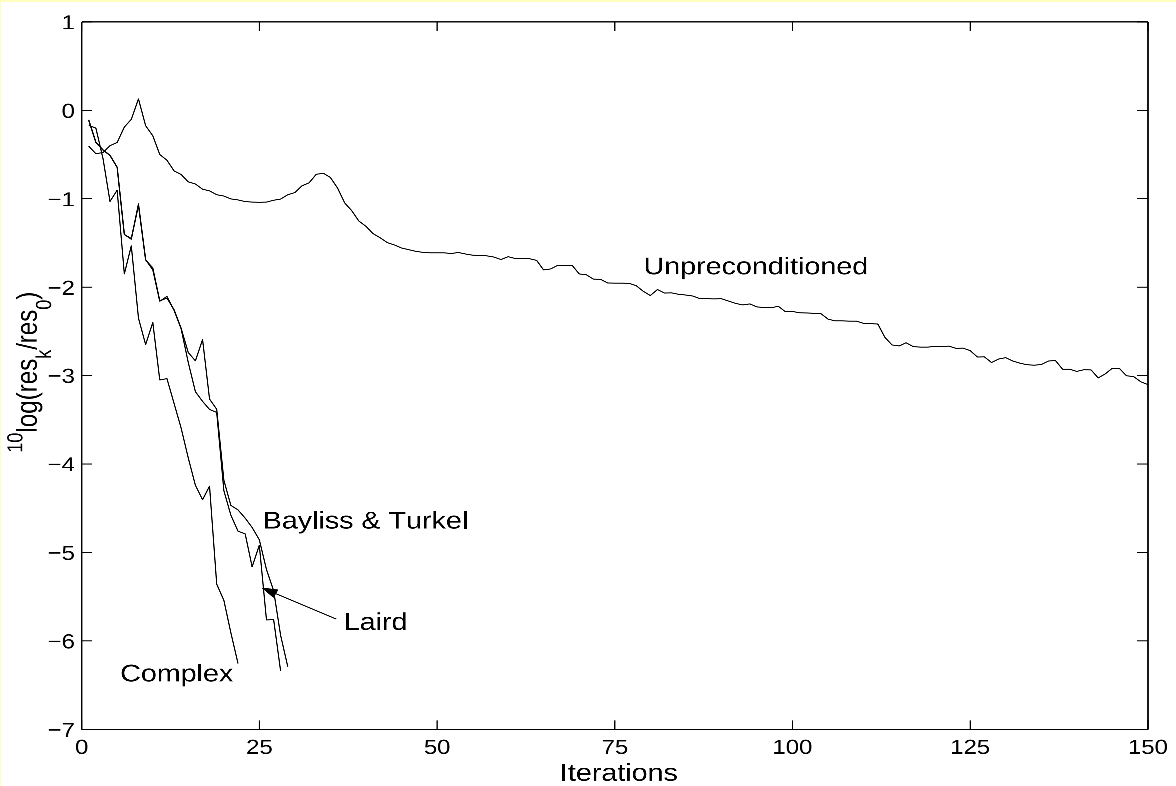
Preconditioner: Shifted-Laplace operator, discretized using the same method as the Helmholtz operator.

| $k$ | ILU(0.01) | $M_0$ | $M_1$ | $M_i$ |
|---|---|---|---|---|
| 5 | 9 | 13 | 13 | 13 |
| 10 | 25 | 29 | 28 | 22 |
| 15 | 47 | 114 | 45 | 26 |
| 20 | 82 | 354 | 85 | 34 |
| 30 | 139 | $> 1000$ | 150 | 52 |

TU Delft

# *Example with constant $k$ in $\Omega$*

Convergence behavior for $k = 10$




TUDelft

# *Example with non-constant $k$ in $\Omega$*

Three-layers problem

$$k = \begin{cases} k_{\text{ref}} & 0 \le y \le 1/3, \\ 1.5k_{\text{ref}} & 1/3 \le y \le 2/3, \\ 2.0k_{\text{ref}} & 2/3 \le y \le 1.0. \end{cases}$$

TUDelft

# *Example with non-constant $k$ in $\Omega$*

Three-layers problem

| | CGNR | | | Bi-CGSTAB | | |
|---|---|---|---|---|---|---|
| $k_{ref}$ | $M_0$ | $M_1$ | $M_i$ | $M_0$ | $M_1$ | $M_i$ |
| 2 | 12 | 12 | 10 | 6 | 7 | 5 |
| 5 | 39 | 31 | 23 | 17 | 15 | 13 |
| 10 | 189 | 88 | 66 | 150 | 56 | 22 |
| 15 | 647 | 175 | 126 | 685 | 113 | 40 |
| 20 | $>$1000 | 268 | 194 | $>$1000 | 177 | 60 |
| 30 | $>$1000 | 502 | 361 | $>$1000 | 344 | 105 |

TUDelft

## 6. Conclusions

- The shifted Laplace operator leads to a new class of preconditioners for the Helmholtz equation.
- Except for $\alpha = 0, \beta = 0$ (Bayliss & Turkel), the eigenvalues of the preconditioned linear system have an upperbound.
- Numerical tests show the effectiveness of the preconditioners
- For small $k$ the Bayliss & Turkel preconditioner is optimal.
- For large $k$ the complex shifted Laplace preconditioner is optimal.

## *Further information/research*

- http://ta.twi.tudelft.nl/nw/users/vuik/pub03.html
- Y.A. Erlangga and C. Vuik and C.W. Oosterlee
  On a class of preconditioners for solving the Helmholtz equation
  Delft University of Technology
  Department of Applied Mathematical Analysis
  Report 03-01
- Current research
  efficient solution of the systems:

$$M_i s = r \text{ in order to compute } s = M_i^{-1} r$$

using inner-outer iteration methods.

TU Delft