On complex shifted Laplace preconditioners for the vector Helmholtz equation

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Householder Symposium XVII Zeuthen, Germany

June 1-6, 2008

Contents

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Financially supported by the Dutch Ministry of Economic Affairs: project BTS01044

1. Introduction

The Helmholtz problem is defined as follows

$$
\begin{aligned}
-\partial_{xx}u - \partial_{yy}u - z_1 k^2(x,y)u = f, &\quad \text{in} \quad \Omega, \\
\text{Boundary conditions} &\quad \text{on} \quad \Gamma = \partial\Omega,
\end{aligned}
$$

where:

- $z_1 = \alpha_1 + \mathrm{i}\beta_1$ and $k(x,y)$ is the wavenumber
- for ”solid” boundaries: Dirichlet/Neumann
- for ”fictitious” boundaries: Sommerfeld $\frac{du}{dn} - \mathrm{i}ku = 0$
- Perfectly Matched Layer (PML)
- Absorbing Boundary Layer (ABL)

Discretization

In general: Finite Difference/Finite Element Methods.

Particular to the present case: 5-point Finite Difference stencil, $\mathcal{O}(h^2)$.

Linear system

$$Ax = b, \;\; A \in \mathbb{C}^{N\times N}, \;\; b, x \in \mathbb{C}^N,$$

Discretization

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Linear system

$$Ax = b, \quad A \in \mathbb{C}^{N\times N}, \quad b, x \in \mathbb{C}^N,$$

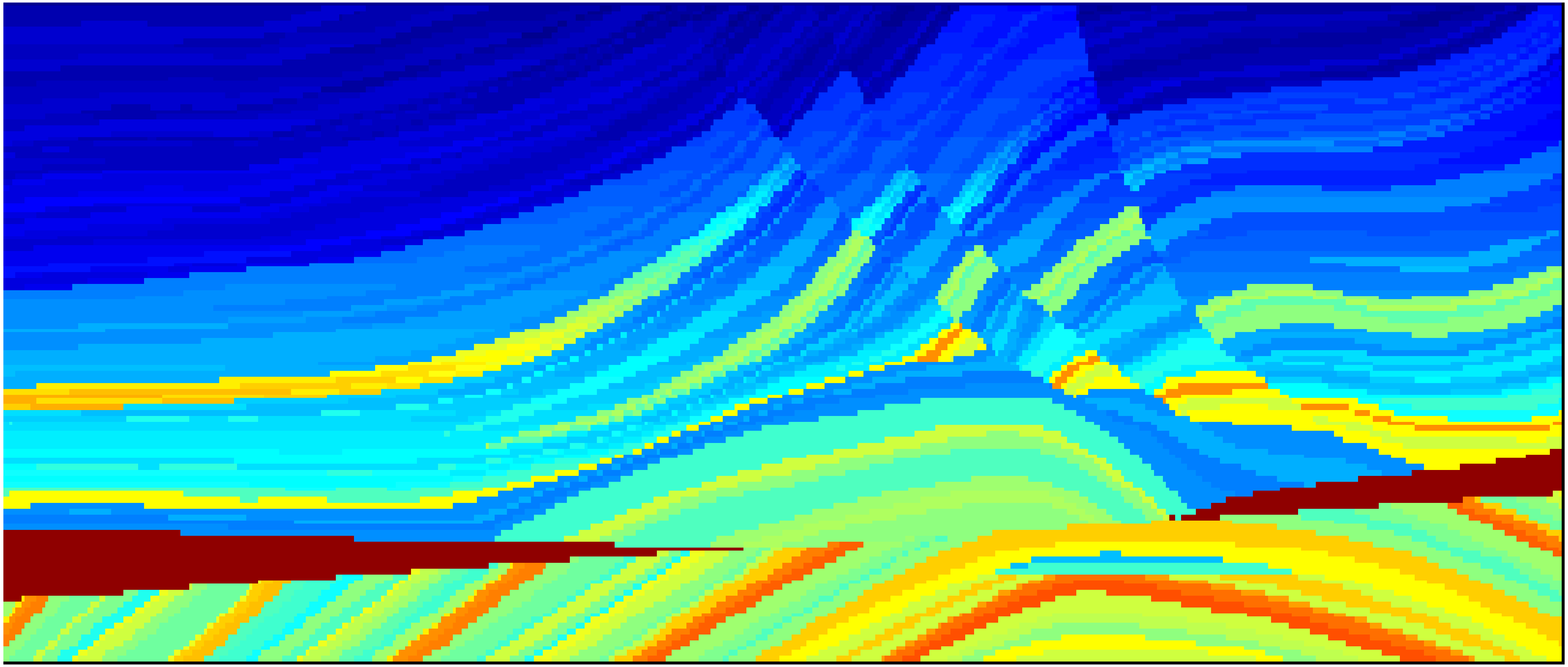
A is a sparse, highly indefinite matrix for practical values of k.
Special property $A = A^T$.

For high resolution a very fine grid is required: $30 - 60$ grid-points per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A$ is extremely large!

Characteristic properties of the problem

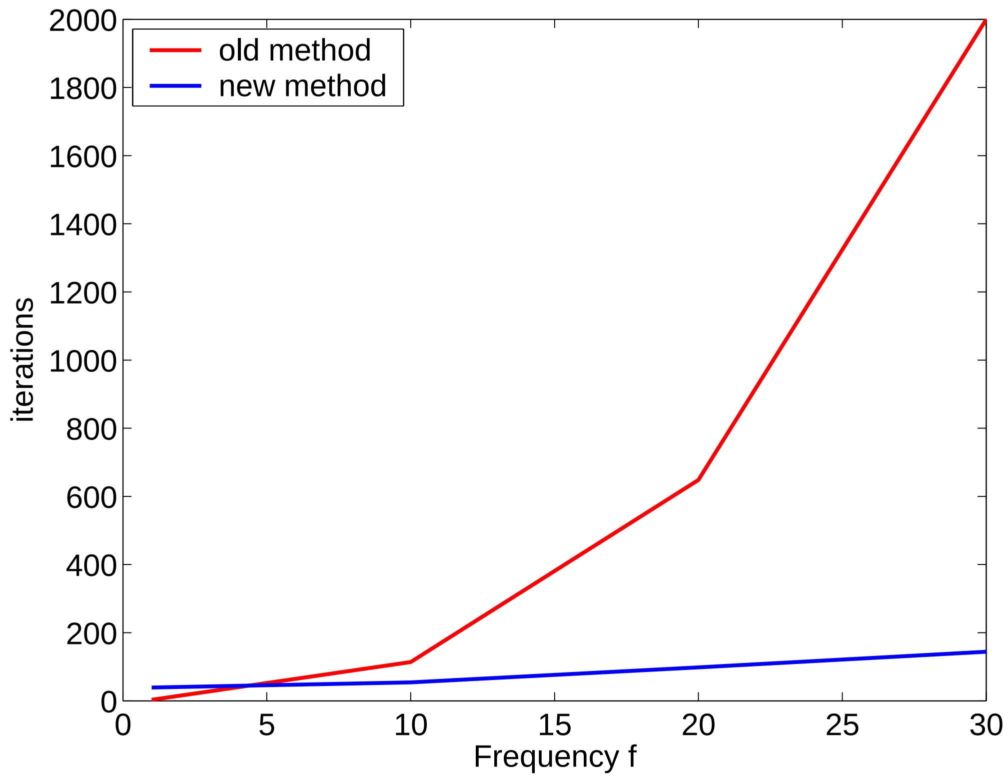
- $A \in \mathbb{C}^{N \times N}$ is sparse
- wavenumber k and grid size N are very large
- wavenumber k varies discontinuously
- real parts of the eigenvalues of A are positive and negative

Application: geophysical survey



Marmousi model (hard)

Application: geophysical survey



Marmousi model (hard)

2. Spectrum of shifted Laplacian preconditioners

Operator based preconditioner P is based on a discrete version of

$$-\partial_{xx}u - \partial_{yy}u - (\alpha_2 + \mathrm{i}\beta_2)k^2(x,y)u = f, \quad \text{in } \Omega.$$

appropriate boundary conditions

Matrix P^{-1} is approximated by an inner iteration process.

$\alpha_2 = 0$	$\beta_2 = 0$	Laplacian	Bayliss and Turkel, 1983
$\alpha_2 = -1$	$\beta_2 = 0$	Definite Helmholtz	Laird, 2000
$\alpha_2 = 0$	$\beta_2 = -1$	Complex	Erlangga, Vuik and
$\alpha_2 = 1$	$\beta_2 = -0.5$	'Optimal'	Oosterlee, 2004, 2006

Spectrum of shifted Laplacian preconditioners

After discretization we obtain the (un)damped Helmholtz operator

$$L - z_1 M,$$

where L and M are SPD matrices and $z_1 = \alpha_1 + \mathrm{i}\beta_1$.

The preconditioner is then given by

$$L - z_2 M,$$

where $z_2 = \alpha_2 + \mathrm{i}\beta_2$ is chosen such that

- systems with the preconditioner are easy to solve,
- the outer Krylov process is accelerated significantly.

Spectrum of shifted Laplacian preconditioners

References: Manteuffel, Parter, 1990; Yserentant, 1988

Since L and M are SPD we have the following eigenpairs

$$Lv_j = \lambda_j M v_j, \text{ where, } \lambda_j \in \mathbb{R}^+$$

The eigenvalues σ_j of the preconditioned matrix satisfy

$$(L - z_1 M)v_j = \sigma_j (L - z_2 M) v_j.$$

Theorem 1
Provided that $z_2 \neq \lambda_j$, the relation

$$\sigma_j = \frac{\lambda_j - z_1}{\lambda_j - z_2} \text{ holds.}$$

Spectrum of shifted Laplacian preconditioners

Theorem 2

If $\beta_2 = 0$, the eigenvalues $\sigma_r + \mathrm{i}\sigma_i$ are located on the straight line in the complex plane given by

$$\beta_1 \sigma_r - (\alpha_1 - \alpha_2)\sigma_i = \beta_1.$$

Spectrum of shifted Laplacian preconditioners

Theorem 2
If $\beta_2 = 0$, the eigenvalues $\sigma_r + \mathrm{i}\sigma_i$ are located on the straight line in the complex plane given by

$$\beta_1 \sigma_r - (\alpha_1 - \alpha_2)\sigma_i = \beta_1.$$

Theorem 3
If $\beta_2 \neq 0$, the eigenvalues $\sigma_r + \mathrm{i}\sigma_i$ are on the circle in the complex plane with center c and radius R:

$$c = \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_2}, \quad R = \left| \frac{z_2 - z_1}{z_2 - \bar{z}_2} \right|.$$

Note that if $\beta_1 \beta_2 > 0$ the origin is not enclosed in the circle.

Spectrum of shifted Laplacian preconditioners

Using Sommerfeld boundary conditions, it impossible to write the matrix as $L - z_1 M$ where, L and M are SPD.

Generalized matrix

$$L + \mathrm{i}C - z_1 M,$$

where L, M, and C are SPD. Matrix C contains Sommerfeld boundary conditions (or other conditions: PML, ABL).

Use as preconditioner

$$L + \mathrm{i}C - z_2 M.$$

Spectrum of shifted Laplacian preconditioners

Suppose

$$(L + \mathrm{i}C)v = \lambda_C M v$$

then

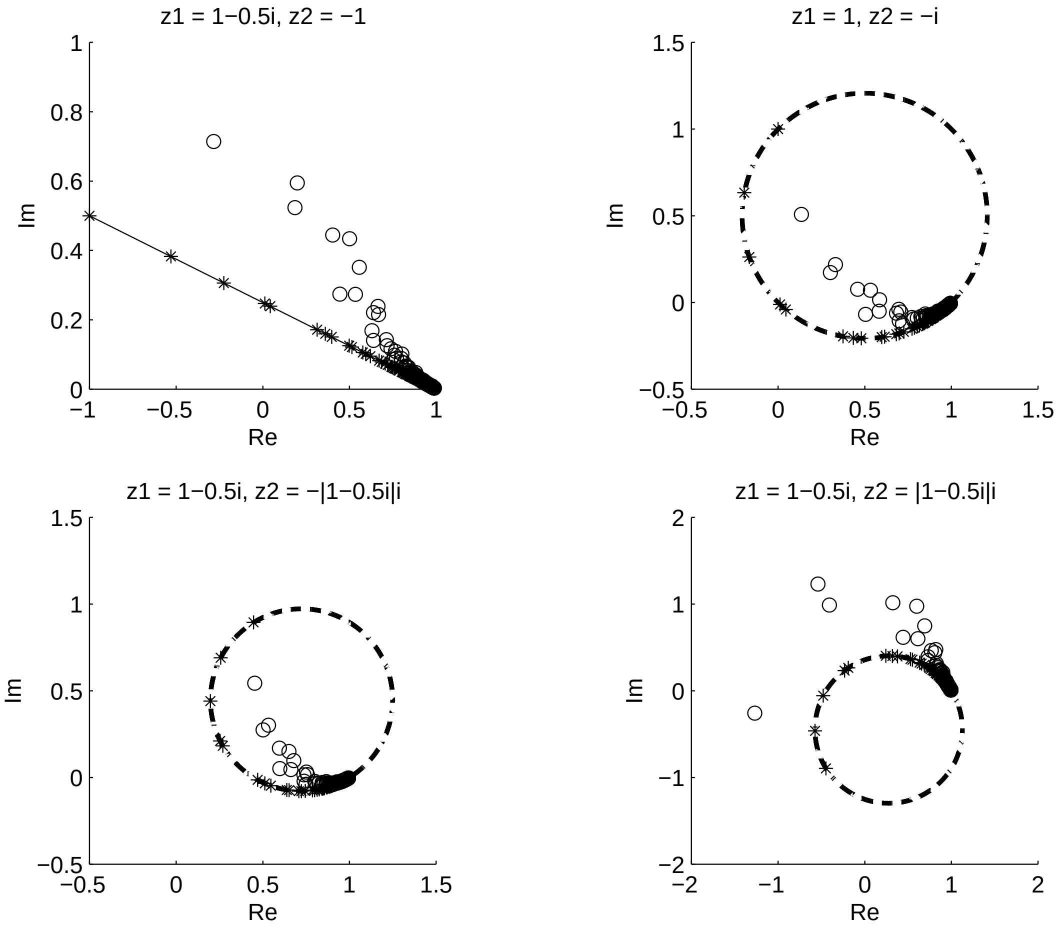
$$(L + \mathrm{i}C - z_1 M)v = \sigma_C (L + \mathrm{i}C - z_2 M)v.$$

Theorem 4

Let $\beta_2 \neq 0$ then the eigenvalues σ_C are in or on the circle with center $c = \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_2}$ and radius $R = \left| \frac{z_2 - z_1}{z_2 - \bar{z}_2} \right|$.

3. Shift with an SPD real part

Motivation: the preconditioned system is easy to solve.



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Optimal choices for z_2?

Damping	Optimal β_2	"optimal" iterations	Minimum iterations
$\beta_1 = 0$	-1	56	54
$\beta_1 = -0.1$	-1.005	42	41
$\beta_1 = -0.5$	-1.118	20	20
$\beta_1 = -1$	-1.4142	13	13

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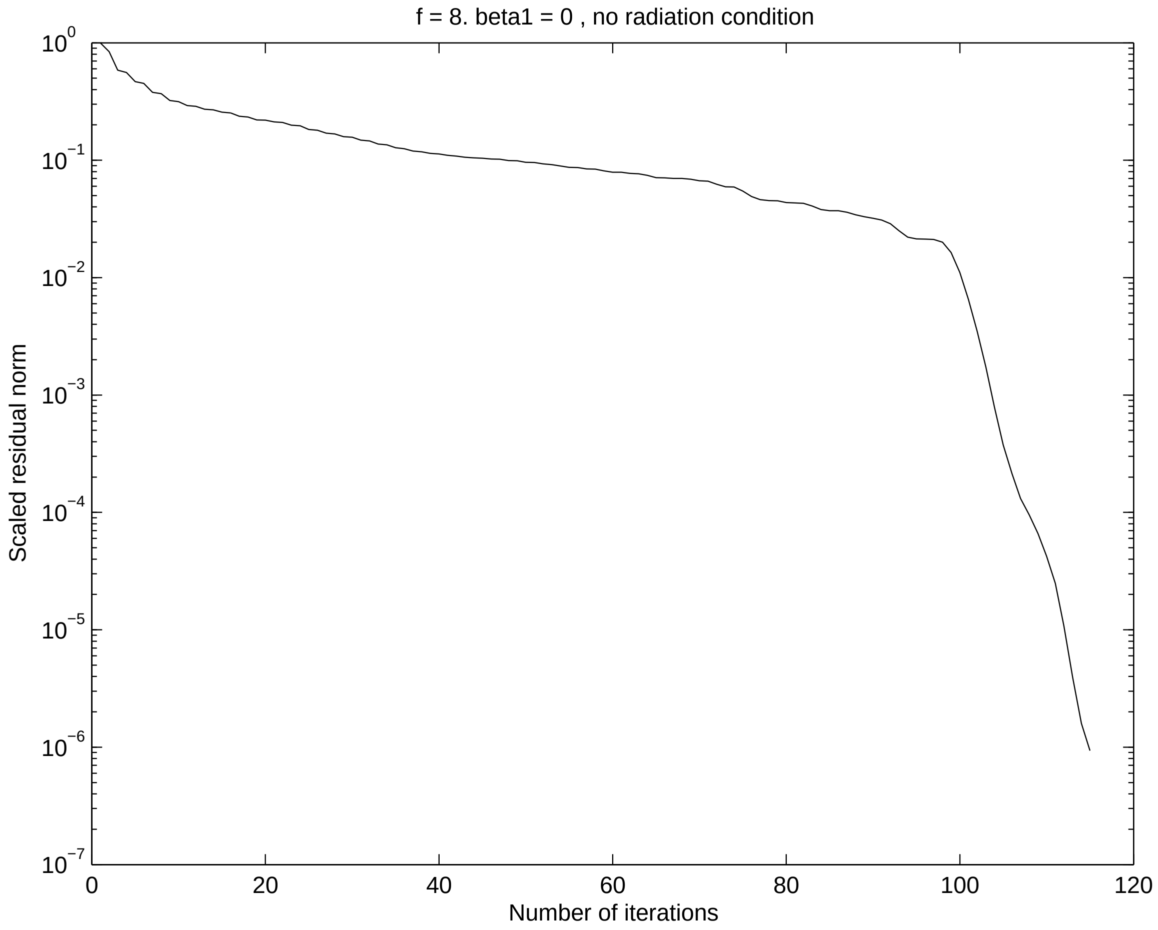
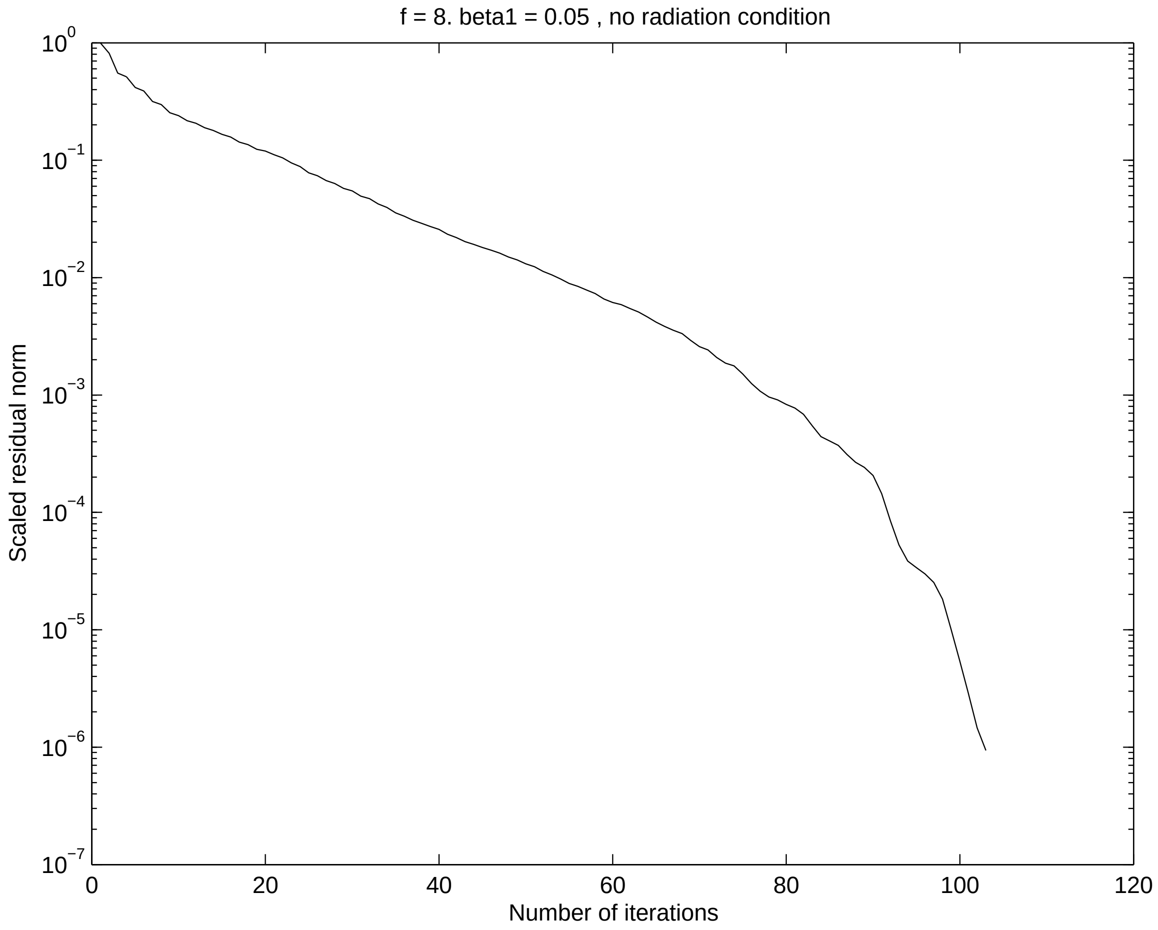
	Number of iterations				
h f	100/2 2	100/4 4	100/8 8	100/16 16	100/32 32
$\beta_1 = 0$	14	25	56	116	215
$\beta_1 = -0.1$	13	22	42	63	80
$\beta_1 = -0.5$	11	16	20	23	23
$\beta_1 = -1$	9	11	13	13	13

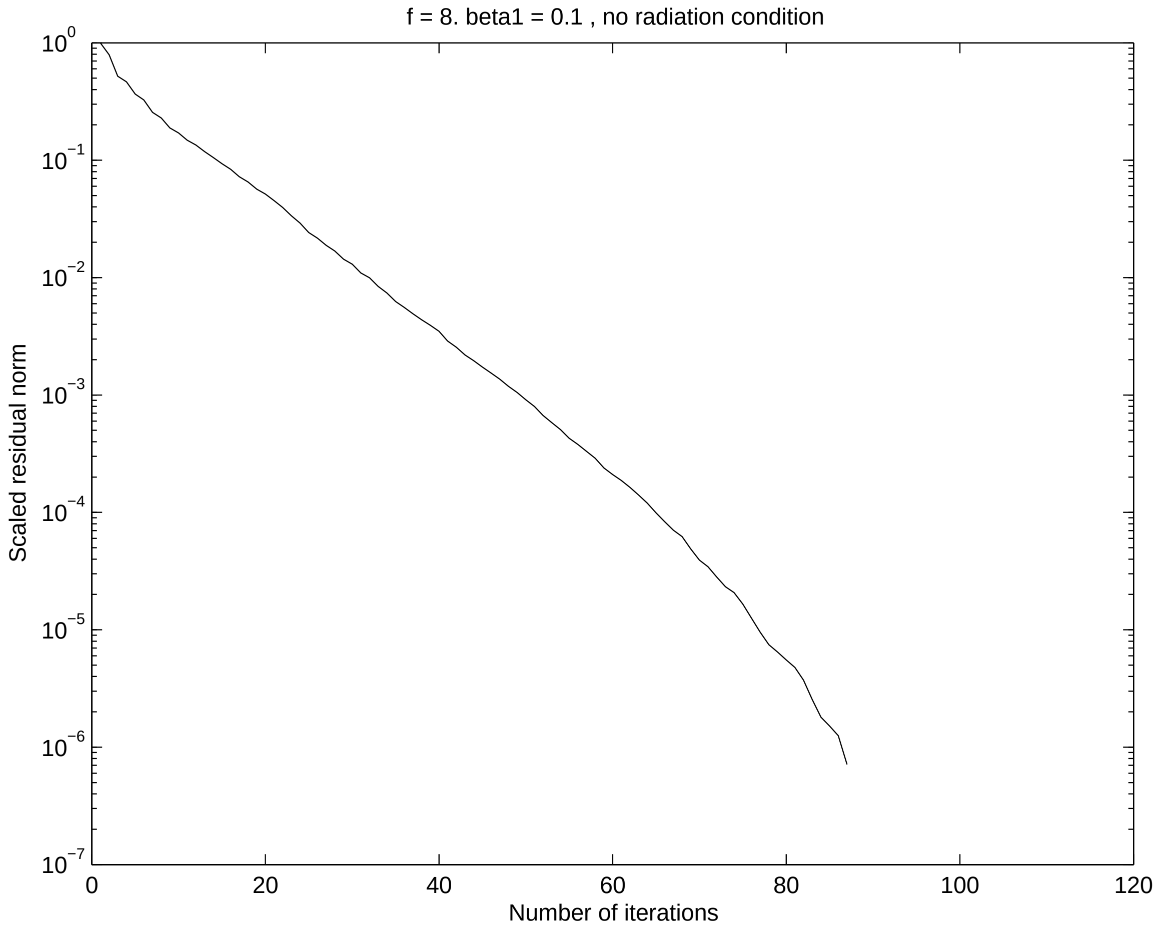
Superlinear convergence of GMRES



Superlinear convergence of GMRES



Superlinear convergence of GMRES



4. General Shifted Laplacian preconditioner

No restriction on α_2

For the outer loop $\alpha_2 = 1$ and $\beta_2 = 0$ is optimal. Convergence in 1 iteration. **But**, the inner loop does not converge with multi-grid (original problem).

However, it appears that multi-grid works well for $\alpha_2 = 1$ and $\beta_2 = -1$

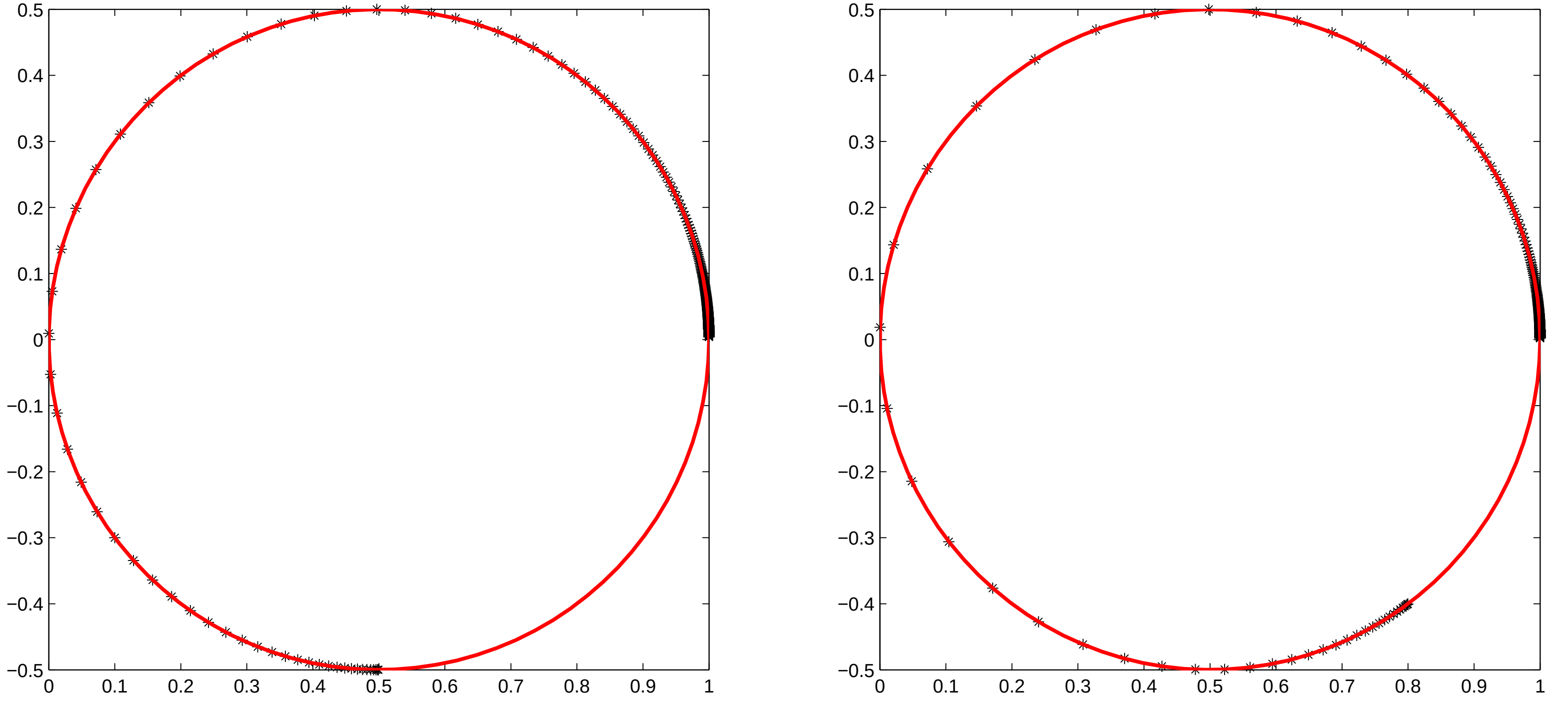
and the convergence of the outer loop is much faster than for the choice

$\alpha_2 = 0$ and $\beta_2 = -1$.

Eigenvalues for Complex preconditioner $k = 100$ *and* $\alpha_2 = 1$

Spectrum is independent of the grid size

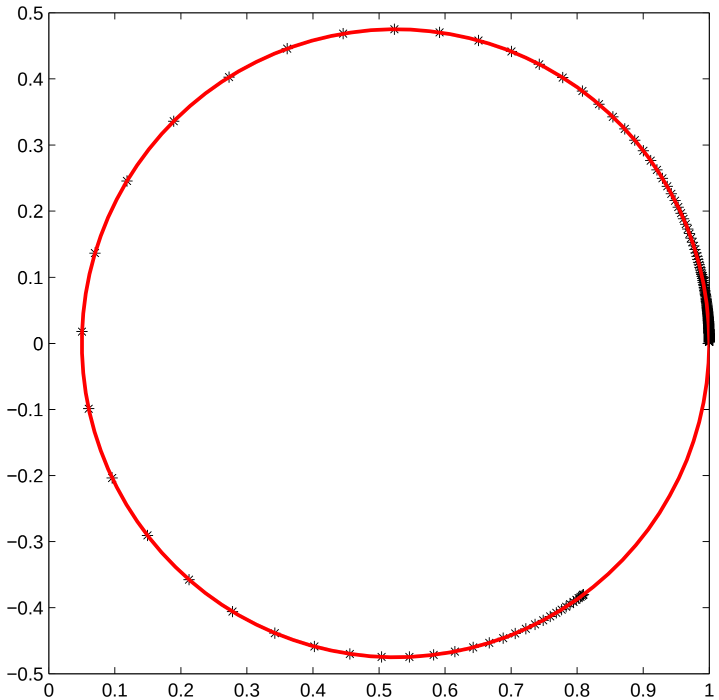
$\beta_2 = -1$ $\qquad\qquad$ $\beta_2 = -0.5$



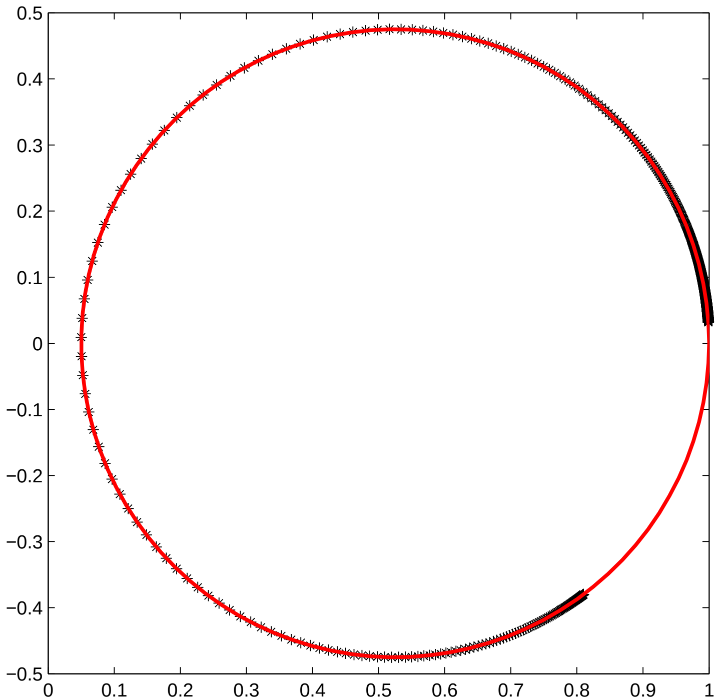
Eigenvalues for $\beta_1 = -0.025$ *(damping) and* $\alpha_2 = -1$, $\beta_2 = -0.5$

Spectrum is independent of the grid size and the choice of k.

$k = 100$

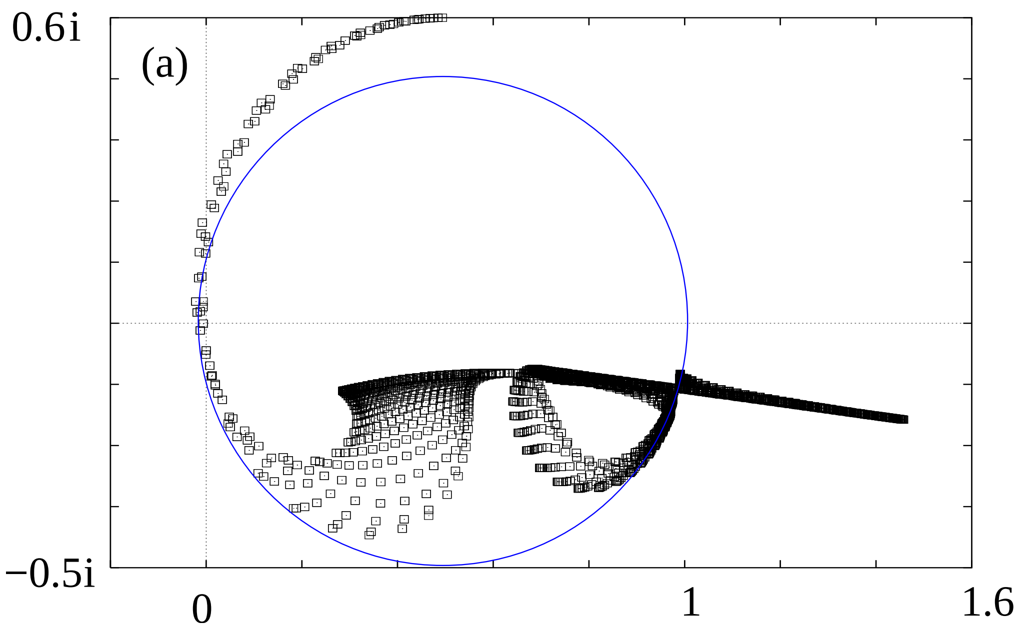


$k = 400$



5. Numerical experiments

Spectrum with inner iteration



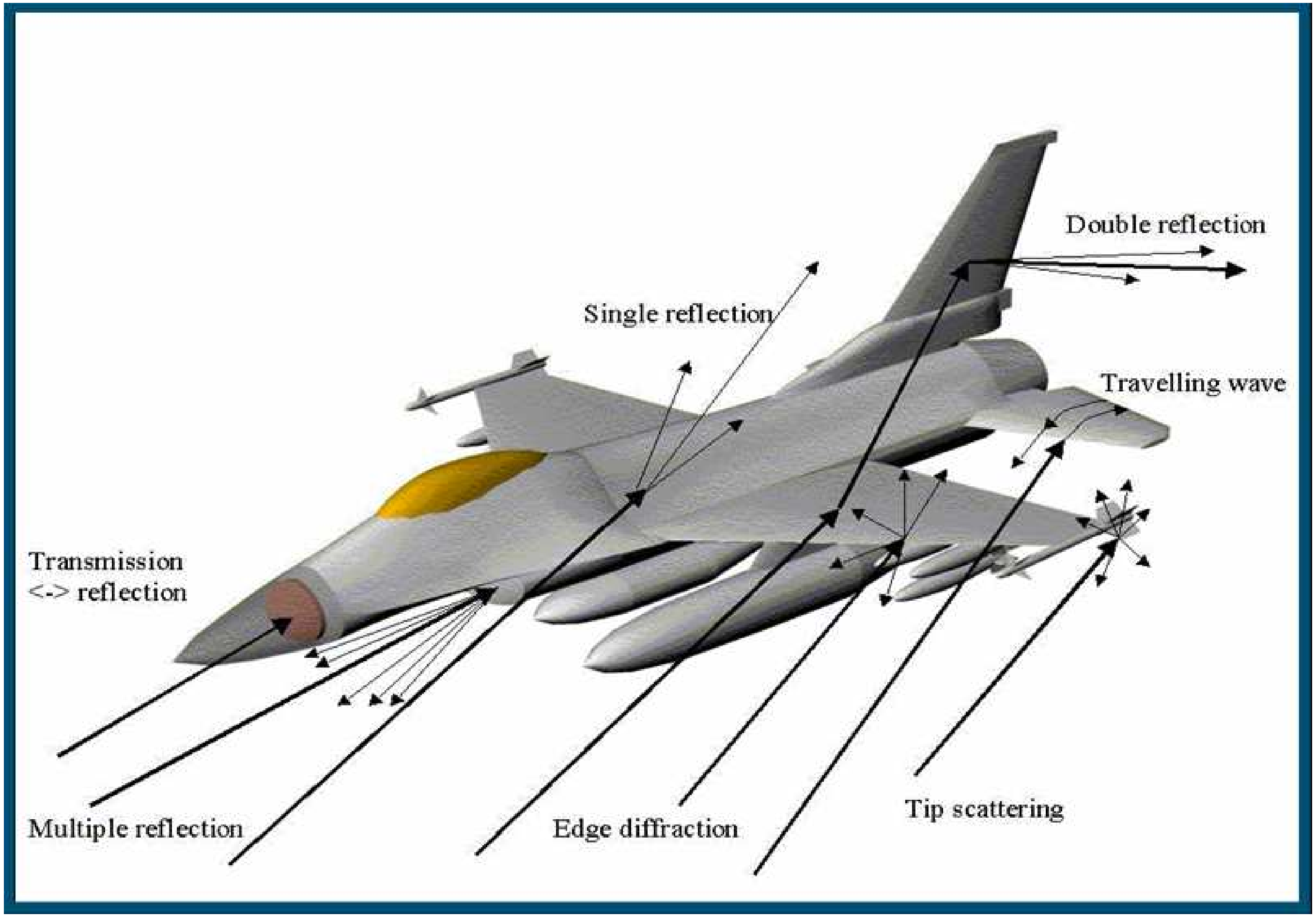
Sigsbee model

$dx = dz = 22.86$ m; $D = 24369 \times 9144$ m^2; grid points 1067×401.

Bi-CGSTAB	5 Hz		10 Hz	
	CPU (sec)	Iter	CPU (sec)	Iter
NO preco	3128	16549	1816	9673
With preco	86	48	92	58

Note: ▶ Without preconditioner, number of iterations $> 10^4$,
▶ With shifted Laplacian preconditioner, only 58 iterations.

6. Fighter radar signature



Jet engine air intake scattering

- Jet engine air intake closed by jet engine compressor fan forms a large and deep open-ended cavity with varying cross section
- typical dimensions: $D \approx 30\lambda$ and $L \approx 200\lambda$ for X-band excitation (10 GHz)

Simulation of large cavity scattering

Different approaches for modelling deep cavity scattering:

- Asymptotic techniques based on ray-tracing and Geometrical/Physical Optics (become inaccurate for $\frac{L}{D} > 3$)
- Modal expansion methods (only for (piecewise) cylindrical cavities)
- Full wave methods based on discretisation of Maxwell equations (no restriction on cavity geometry but expensive for electrically large cavities)

Only full wave method offers flexibility to accurately model the actual geometry: FE for vector wave equation

Iterative solution of the linear system

Use preconditioned Krylov method to solve

$$-\nabla \times \nabla \times \mathbf{E} - k_0^2 E = 0$$

Using as a preconditioner a single multigrid cycle of

$$-\nabla \times \nabla \times \mathbf{E} - \tilde{k}_0^2 E = 0, \tilde{k}_0^2 = (\alpha_2 + i\beta_2)k_0^2$$

- For exploratory study we used a Krylov method for the preconditioner solve, with fixed residual reduction factor or fixed number of iterations.
- Analyse different strategies for handling the discretised BIE
- Find optimal values of α_2 and β_2.

Iterative solution of the linear system

Use preconditioned GCR to solve

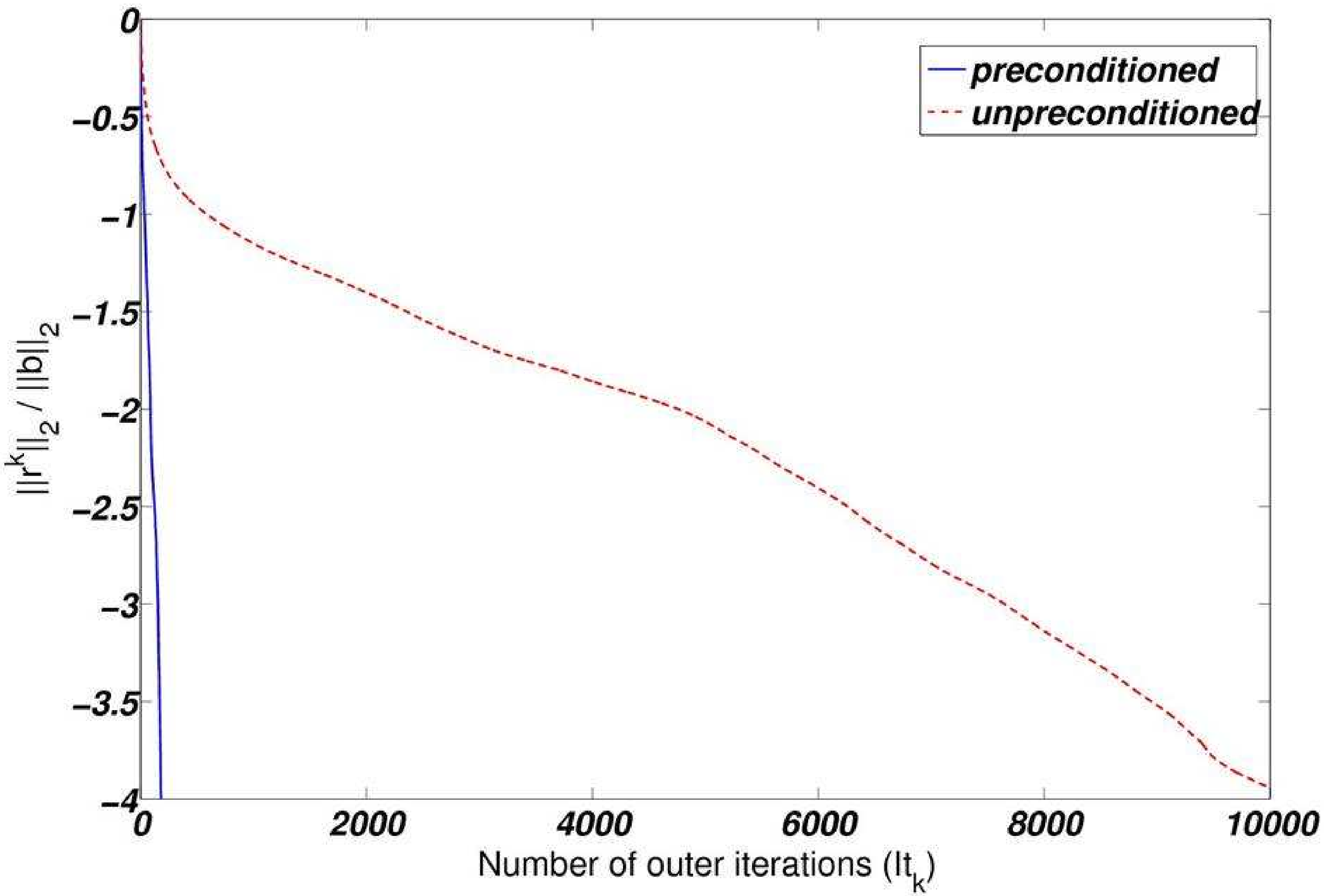
$$\begin{pmatrix} A_{11}(k_0) & A_{12}(k_0) \\ A_{21}(k_0) & A_{22}(k_0) \end{pmatrix} \begin{pmatrix} E_i \\ E_a \end{pmatrix} = \begin{pmatrix} 0 \\ b(k_0) \end{pmatrix}$$

Using the following block preconditioner:

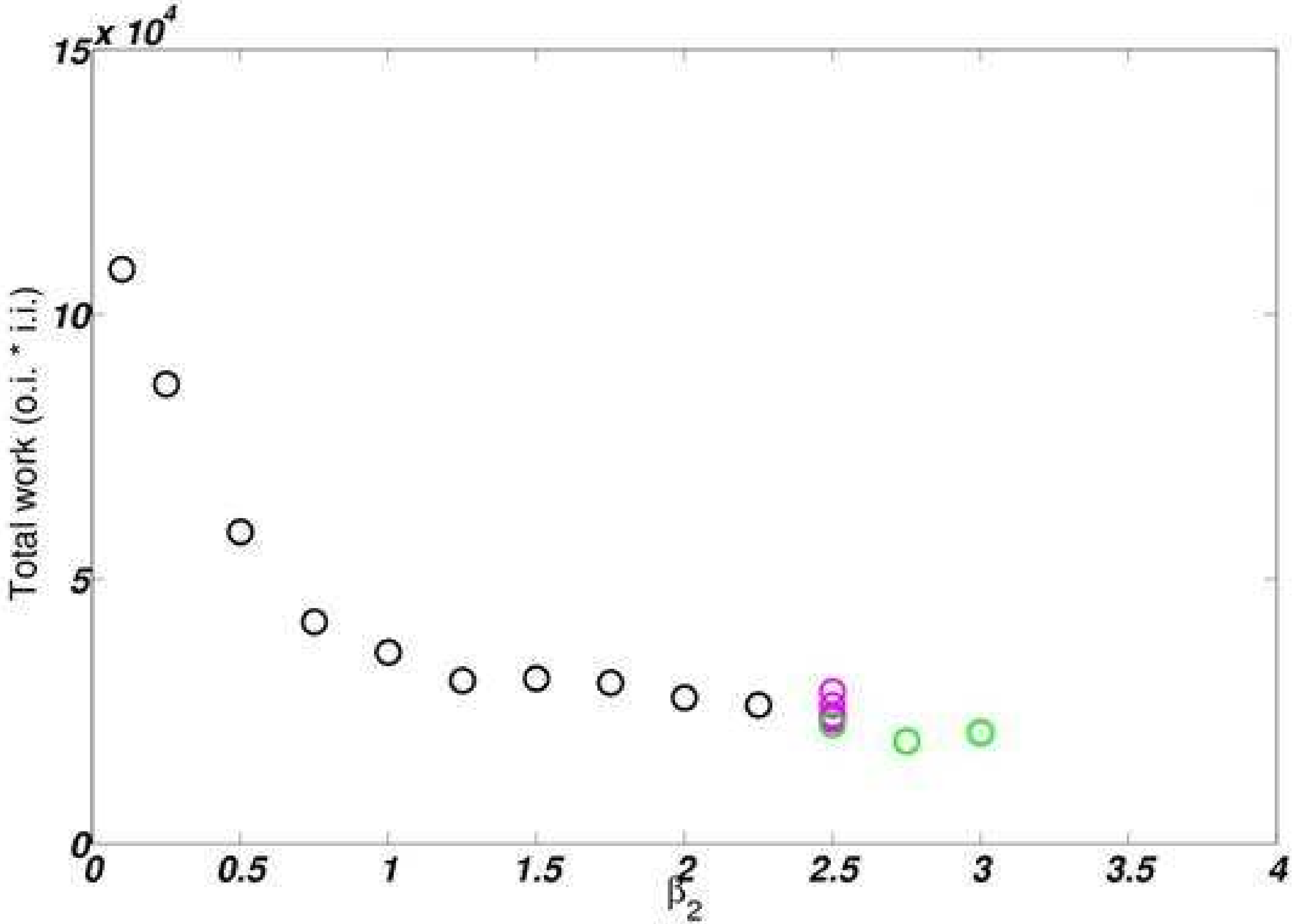
$$\begin{pmatrix} A_{11}(\tilde{k}_0) & A_{12}(k_0) \\ 0 & A_{22}(k_0) \end{pmatrix} \begin{pmatrix} s_i \\ s_a \end{pmatrix} = \begin{pmatrix} r_i \\ r_a \end{pmatrix}$$

- Matrix-free formulation of $A_{11}(k_0)$ for large problems
- Solve $A_{11}(\tilde{k}_0)s_i = r_i - A_{12}(k_0)s_a$ with GCR for fixed ϵ
- Precompute $(LU)_{A_{22}}$ and solve $(LU)_{A_{22}}s_a = r_a$

Iterative solution of the linear system



Iterative solution of the linear system



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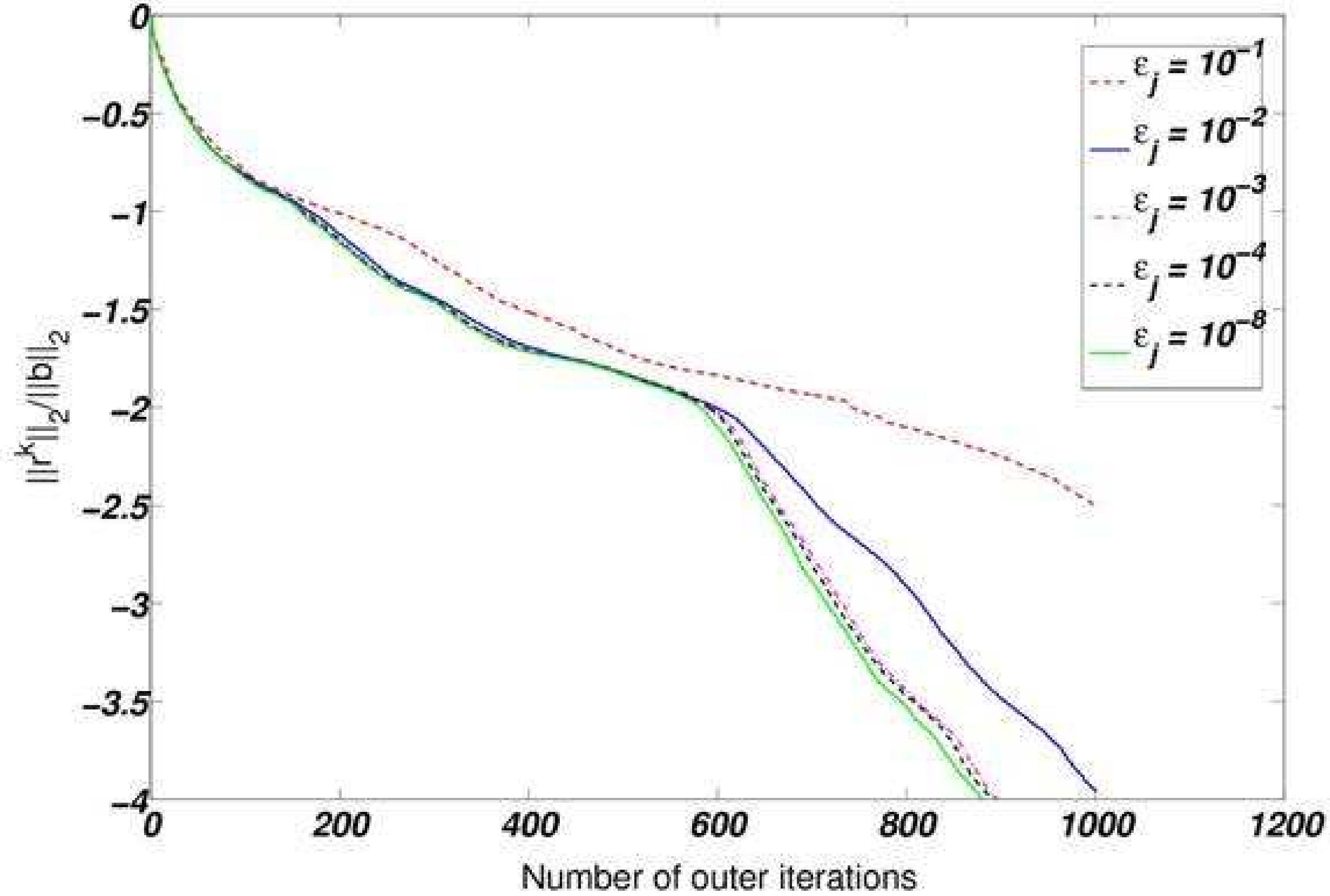
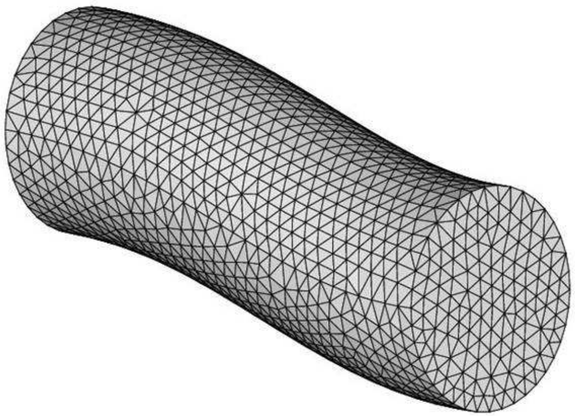
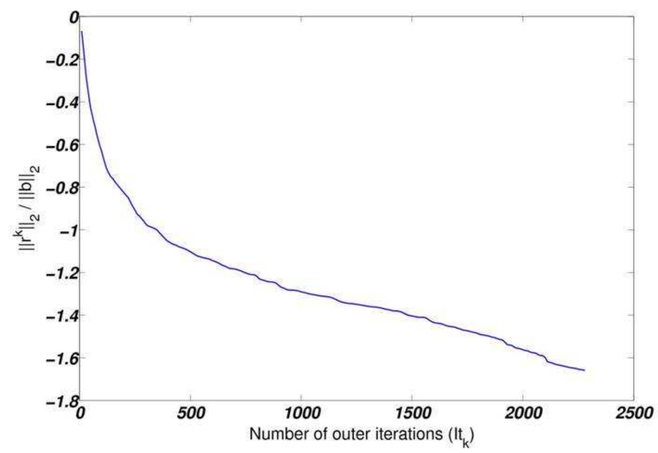
Iterative solution of the linear system



Iterative solution of the linear system





Future research: Multigrid solve of preconditioner essential for larger problems:

- CPU time in preconditioner solve
- Possibility to use short recurrence Krylov method.

7. Conclusions

- The shifted Laplacian operator leads to robust preconditioners for the Helmholtz equations with various boundary conditions.
- For real shifts the eigenvalues of the preconditioned operator are on a straight line.
- For complex shifts the eigenvalues of the preconditioned operator are on a circle.
- The proposed preconditioner (shifted Laplacian + multi-grid) is independent of the grid size and linearly dependent of k.
- With physical damping the proposed preconditioner is also independent of k.
- For the FEM vector Helmholtz problem a good multigrid method is needed for the shifted preconditioner.

Further information/research

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