Fast solvers for seismic problems

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Contents

The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x,y) - k^2(x,y)\mathbf{u}(x,y) = \mathbf{g}(x,y) \quad \text{in } \Omega$$

$\mathbf{u}(x,y)$ is the pressure field,
$\mathbf{k}(x,y)$ is the wave number,
$\mathbf{g}(x,y)$ is the point source function and
Ω is domain bounded by Absorbing boundary conditions

$$\frac{\partial \mathbf{u}}{\partial n} - \iota \mathbf{u} = 0$$

n is normal direction to respective boundary.

Problem description

- Second order Finite difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4-k^2h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $Au = g$: properties

  Sparse & complex valued

  Symmetric & Indefinite for large k

- Is traditionally solved by Krylov subspace method, they exploit the sparsity.

Preconditioning

- ILU and variants
- From Laplace to complex shifted Laplace preconditioner (2005)
- Shifted Laplace preconditioner (SLP)

$$M := -\Delta \mathbf{u} - (\beta_1 - \iota\beta_2)k^2\mathbf{u}$$

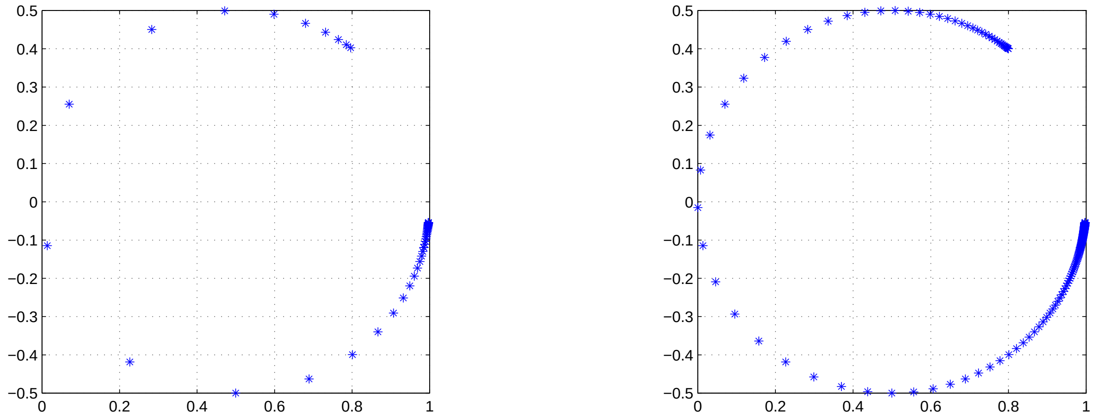
- Results shows: $(\beta_1, \beta_2) = (1, 0.5)$ is shift of choice
- What does SLP do ? ?

Shifted Laplace Preconditioner

- Introduces damping, Multigrid approximation
- Norm of spectrum of preconditioned operator bounded above by 1
- Spectrum goes near to zero, as k increases.

Spectrum of $M^{-1}(1, 0.5)A$ for

$k = 30$ and $k = 120$



Some Results at a Glance

Number of GMRES iterations. Shifts in preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**10**	17	28	44	70	13/14
$n = 64$	10	**17**	28	36	45	173/163
$n = 96$	10	17	**27**	35	43	36/97
$n = 128$	10	17	27	**35**	43	36/85
$n = 160$	10	17	27	35	**43**	25/82
$n = 320$	10	17	27	35	42	**80**

Impact of shifted Laplace preconditioner

Shell

PHILIPS

Impact of shifted Laplace preconditioner

Schlumberger

and many others ....

Impact of shifted Laplace preconditioner

I AHMAD ERLANGGA ON MULTIGRID BASED PRECONDITIONER FOR HETEROGENEOUS HELMHOLTZ BLEMS

RGING RESEARCH FRONT COMMENTARY, OCTOBER 2010

ARTICLE: A NOVEL MULTIGRID BASED PRECONDITIONER FOR HETEROGENEOUS HELMHOLTZ PROBLEMS

Authors: **Erlangga, Y**;Oosterlee, C;Vuik, C
Journal: SIAM J SCI COMPUT, 27 (4): 1471-1492, 2006
Addresses: Delft Univ Technol, Fac Elect Engn Math & Comp Sci, Delft, Netherlands.
Delft Univ Technol, Fac Elect Engn Math & Comp Sci, Delft, Netherlands.

Yogi Ahmad Erlangga talks with *ScienceWatch.com* and answers a few questions about this month's Emerging Research Front paper in the field of Mathematics.

SW: **Why do you think your paper is highly cited?**

Every year for the last five years, I've reviewed around two papers related to the work in the paper in question. There are many papers, which I became aware of over time. So, there are indeed citations. But, to answer your question, I first have to say that I had never thought that it would be highly cited.

One expert in iterative methods said that the problem I solved in the paper is an example that is often avoided because of its inherent difficulties. As it is true, it is so unfortunate because the underlying mathematical model

Deflation improves!!!

Number of GMRES iterations. Shifts in preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 64$	4/10	**6/17**	8/28	12/36	18/45	173/163
$n = 96$	3/10	5/17	**7/27**	9/35	12/43	36/97
$n = 128$	3/10	4/17	6/27	**7/35**	9/43	36/85
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Erlangga and Nabben, 2008

with / without deflation.

Deflation: Definiton

For any deflation deflation subspace matrix

$$Z \in R^{n \times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rank Z = r$$

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \ \text{and} E = Z^T AZ$$

Solve $PAu = Pg$ preconditioned by M^{-1} or $M^{-1}PA = M^{-1}Pg$
For e.g. say,

$$\textbf{spec}(A) = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n\}$$

and if Z is matrix with columns as the r eigenvectors then

$$\textbf{spec}(PA) = \{0, ..., 0, \lambda_{r+1}, ...\lambda_n\}$$

We use multigrid inter-grid transfer operator (Prolongation) as deflation matrix.

Deflation

Setting $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

$$P = I - AQ, \quad \text{with} \quad Q = I_h^{2h} E^{-1} I_{2h}^h \; \text{and} E = I_{2h}^h A_h^{2h}$$

where

P can be read as coarse grid correction and

Q the coarse grid operator

Fourier Analysis

Dirichlet boundary conditions for analysis.

With above deflation,

$$\textbf{spec}(PM^{-1}A) = f(\beta_1, \beta_2, k, h, l)$$

is a complex valued function.

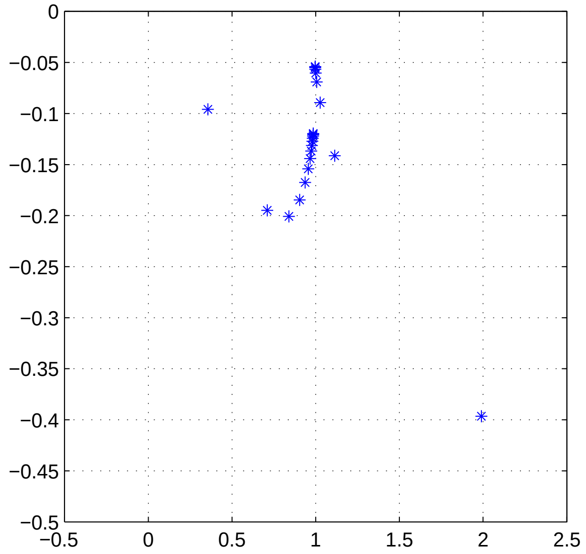
Setting $kh = 0.625$,

- Spectrum of $(I - PM^{-1}A)$ with shifts $(1, 0.5)$ near zero is wrapped and clustered around 1 with few outliers.
- Spectrum remains almost same, when imaginary shift is varied from 0.5 to 1.

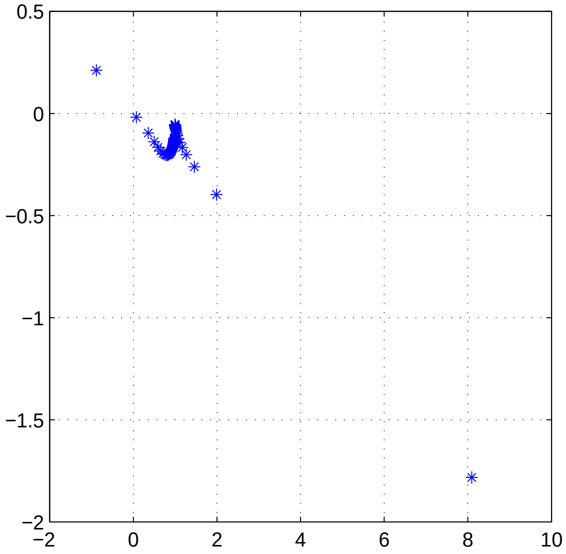
Fourier Analysis

Analysis shows spectrum clustered around 1 with few outliers.

$k = 30$



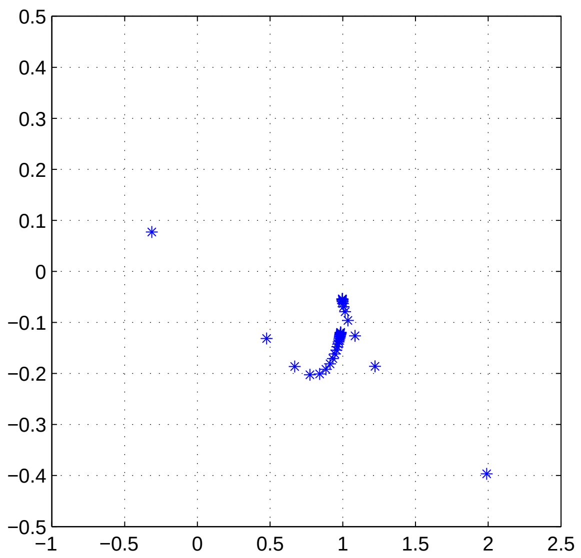
$k = 120$

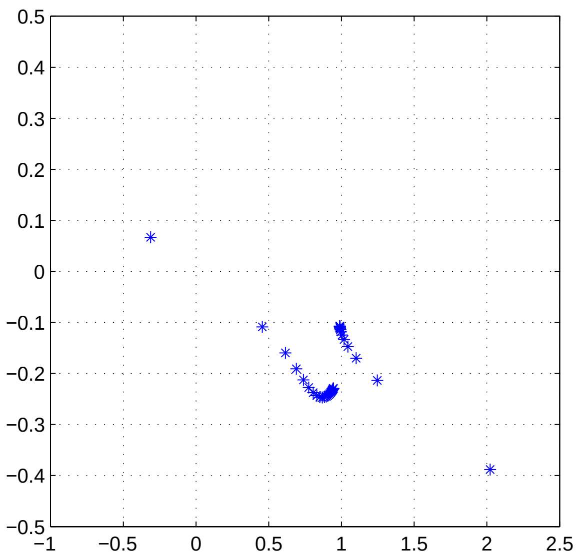


Fourier Analysis

Analysis tells increase in imaginary shift does not change spectrum.

$(\beta_1, \beta_2) = (1, 0.5)$ $(\beta_1, \beta_2) = (1, 1)$



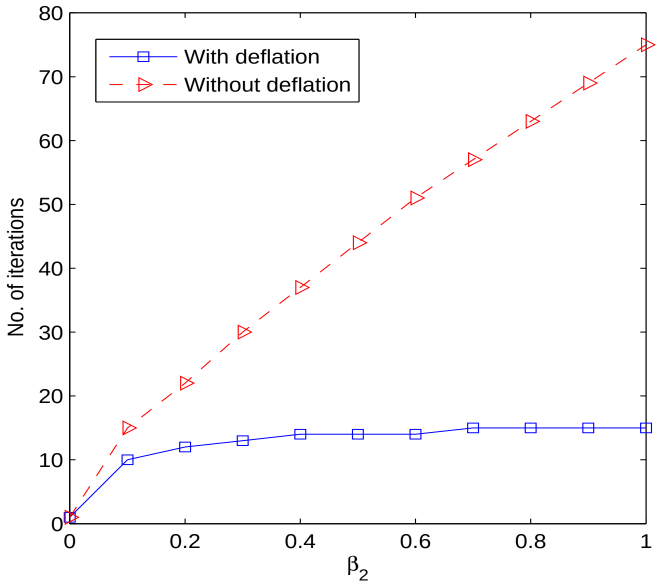


Numerical results

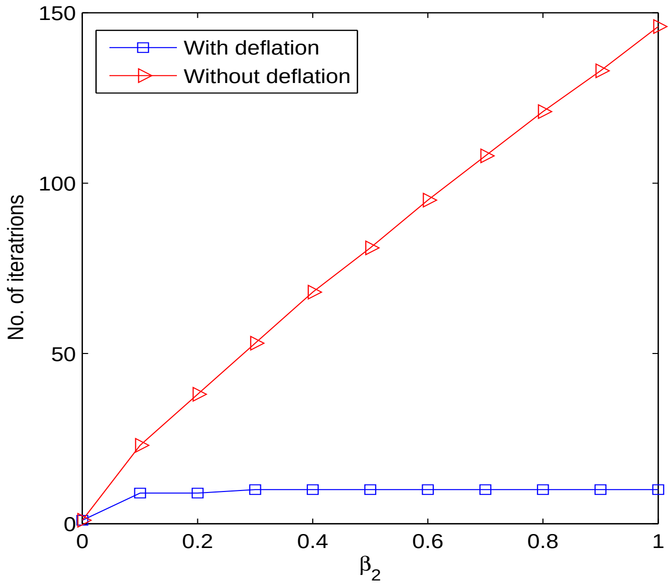
Sommerfeld boundary conditions are used for test problem.

Increase in imaginary shift in SLP ??

Constant wavenumber problem



Wedge problem



Numerical results

Number of GMRES iterations with/without deflation. Shifts in preconditioner are $(1, 0.5)$

Grid	$k=10$	$k=20$	$k=30$	$k=40$	$k=50$	$k=100$
$n=32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n=64$	4/10	**6/17**	8/28	12/36	18/45	173/163
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$n=160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n=320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Numerical results

Number of GMRES iterations with/without deflation to solve a Wedge problem. Shifts in preconditioner are $(1, 0.5)$

Grid	$freq = 10$	$freq = 20$	$freq = 30$	$freq = 40$	$freq = 50$
74×124	**7/33**	20/60	79/95	267/156	490/292
148×248	5/33	**9/57**	17/83	42/112	105/144
232×386	5/33	7/57	**10/81**	25/108	18/129
300×500	4/33	6/57	8/81	**12/105**	18/129
374×624	4/33	5/57	7/80	9/104	**13/128**

Numerical results

Number of GMRES outer-iterations in multilevel algorithm.

$(\beta_1, \beta_2) = (1, 0.5)$

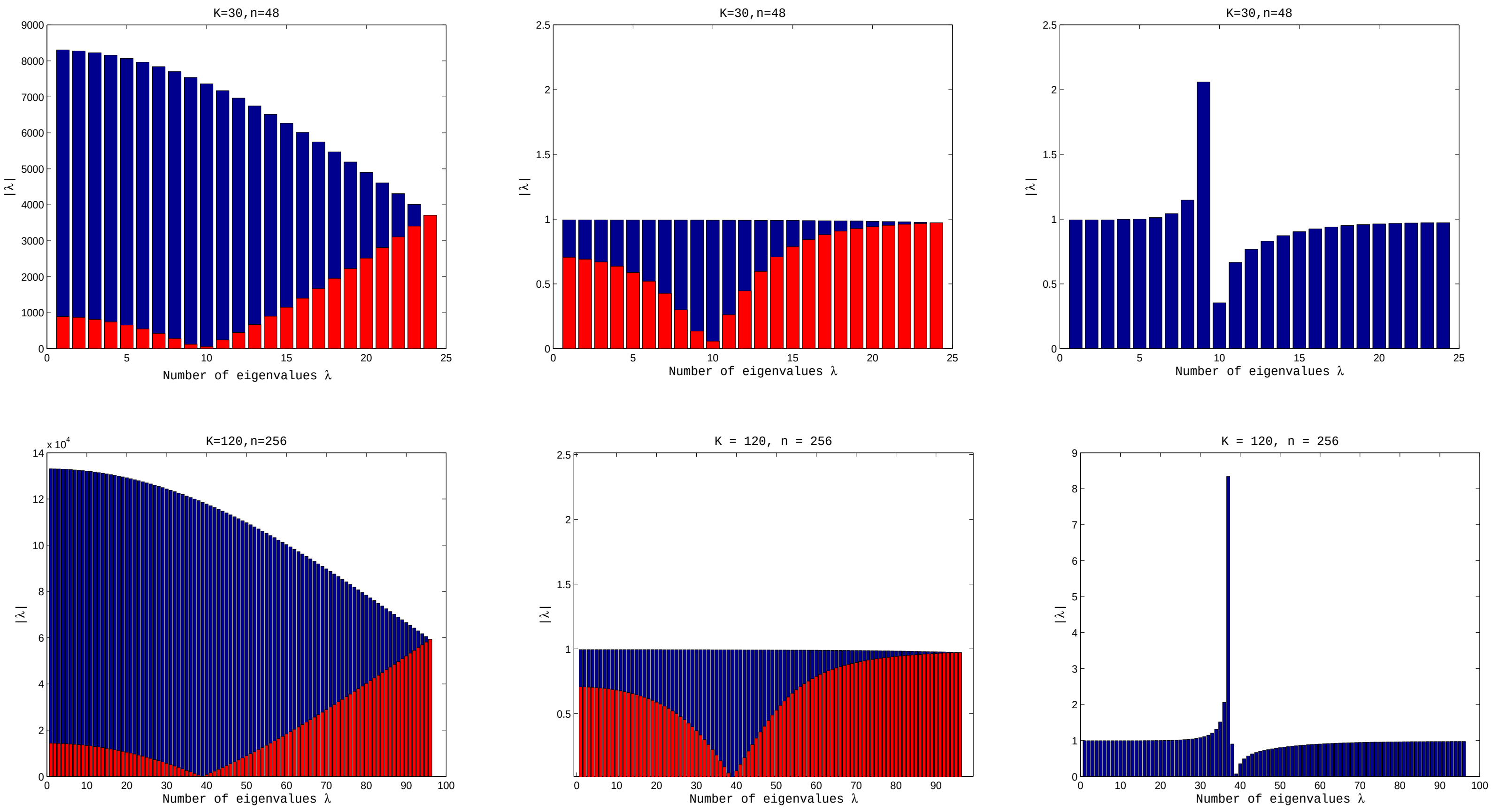
$kh = .3125$ or 20 gp/wl

and MG Vcycle(1,1) for SLP

Grid	$k = 10$	$k = 20$	$k = 40$	$k = 80$	$k = 160$
MLMGV(4,2,1)	9	11	16	27	100+
MLMGV(6,2,1)	9	10	14	21	47
MLMGV(8,2,1)	9	10	13	20	38
MLMGV(8,3,2)	9	10	13	19	37

Fourier Analysis

Spectrum of A, $M^{-1}A$ and $PM^{-1}A$ (from left to right) in bar-graph.



Implementation on Multiple GPU's

Bi-CGSTAB preconditioned by shifted Laplace multigrid method.
Equation solved in preconditioner is

$$-\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} - (\beta_1 - \beta_2 i)k^2\phi = g, \qquad (1)$$

$\beta_1, \beta_2 \in \mathbb{R}$, with the same boundary conditions as the original problem.
Multigrid components:

- Matrix-dependent prolongation (2D: de Zeeuw, 1990, 3D: Zhebel, 2006)
- Standard restriction
- Multi-coloured Gauss-Seidel as a smoother

Preconditioner is computed in single precision.

Little-Green Machine

Little-Green Machine

20 general computing nodes

- 2 Intel quadcore E5620
- 24 GB RAM
- 2 TB disk
- 2 NVIDIA GTX480

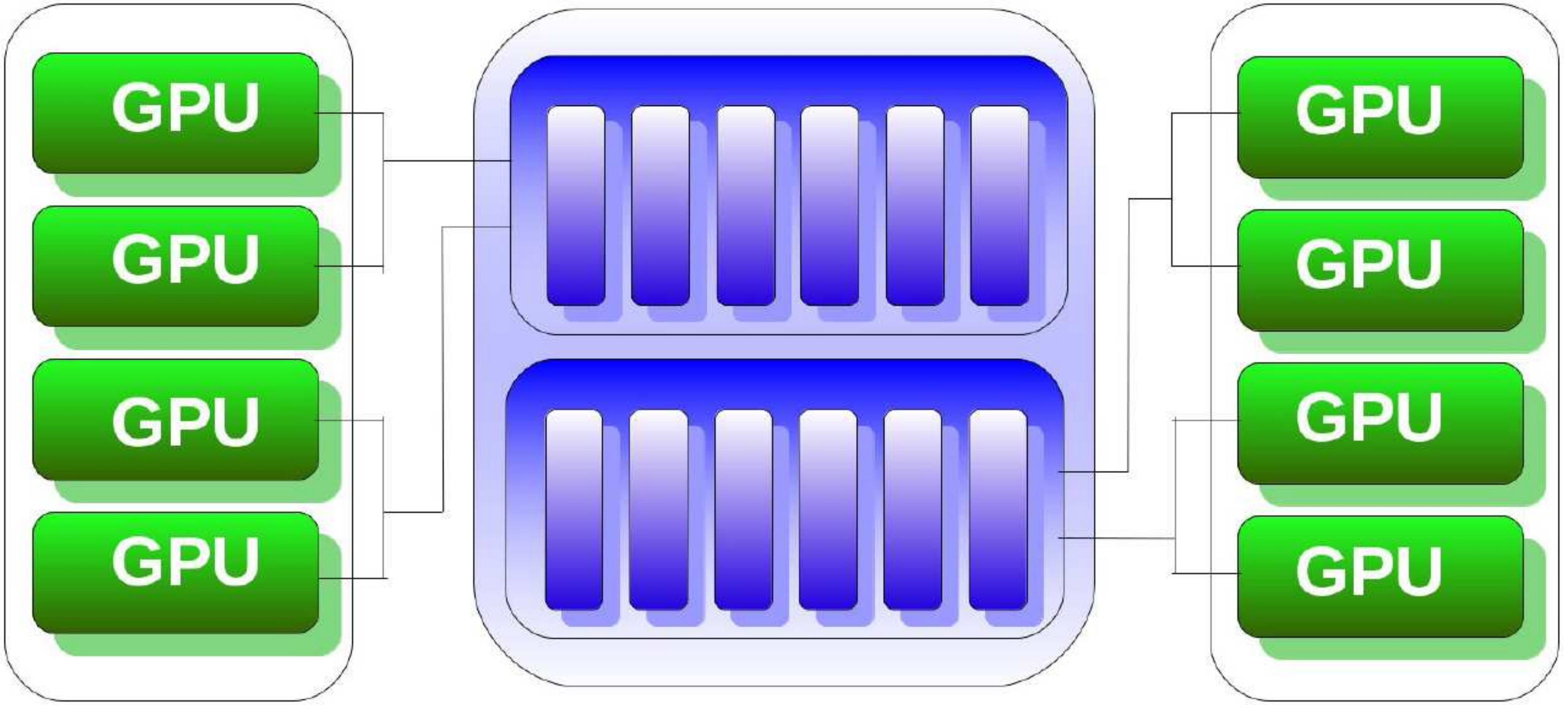
Funded by

- University of Leiden
- NWO
- TU Delft
- KNMI

NVidia Computer

8 GPUs, each GPU has 448 cores, 3 GB RAM

12 cores (2 Westmeres), 48 GB RAM



Gauss-Seidel Smoother

Four color Gauss-Seidel

Size	Time 8-cores (ms)	Time GPU (ms)	CPU/GPU
10,000	4	0.6	**7**
100,000	23.4	0.8	**29**
1,000,000	164.5	2.5	**66**
5,000,000	625.9	10.3	**61**
20,000,000	3733.9	39.4	**95**

Multi-GPU Approach

1. Data-parallel approach (e.g. vector operations on multi-GPU)
   (a) Relatively easy to implement
   (b) CPU→GPU→CPU data transfer
2. Split of the algorithm (e.g. solver on one GPU, preconditioner on the another one)
   (a) No or little data transfers
   (b) Find the best way to split the algorithm
3. Domain-Decomposition approach (e.g. each domain on a different GPU)
   (a) Exchange of halos (still data transfer)
   (b) Can affect convergence of the preconditioned method

Multi-GPU Issues

- Limited GPU memory size so need multiple GPUs for large problems.
- Efficient memory reusage to avoid allocation/deallocation, e.g. pool of GPU-vectors.
- Limit communications CPU→GPU and GPU→CPU.
- Each GPU need separate texture reference.
- Cublas vectors limited to 512 MB.

Bi-CGSTAB

Timings for Bi-CGSTAB, single precision

n	12-cores	1 GPU	**Speedup**	8-GPU	**Speedup**
5,000,000	24 s	0.8 s	**29.8**	2.3 s	**10.5**
15,000,000	82 s	2 s	**38.1**	5.8 s	**14.2**
100,000,000	395 s	-	-	28.6 s	**13.8**

Bi-CGSTAB preconditioned by shifted La

Wedge problem, size $350 \times 350 \times 350 \approx 43,000,000$ unknowns

	Bi-CGSTAB (DP)	Preconditioner (SP)	Total
12-cores	94 s	690 s	784 s
1 GPU	13 s	47 s	60 s
Speedup CPU/GPU	**7.2**	**14.7**	**13.1**
8 GPUs	83 s	86 s	169 s
Speedup CPU/GPUs	**1.1**	**7.9**	**4.6**
2 GPUs+split	12 s	38 s	50 s
Speedup CPU/GPU	**7.8**	**18.2**	**15.5**

Conclusions

- Parameter independent scheme
- Numerical results confirms analysis.
- Flexibility to increase imaginary shift, when deflation is combined with SLP.
- Without deflation, when imaginary shift is increased in SLP, spectrum remains bounded above 1, but lower part moves to zero.
- Further Multilevel scheme, applying similarly for coarse problem in deflation.

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Thank You for Your Attention