A fast solver for the Navier-Stokes equations

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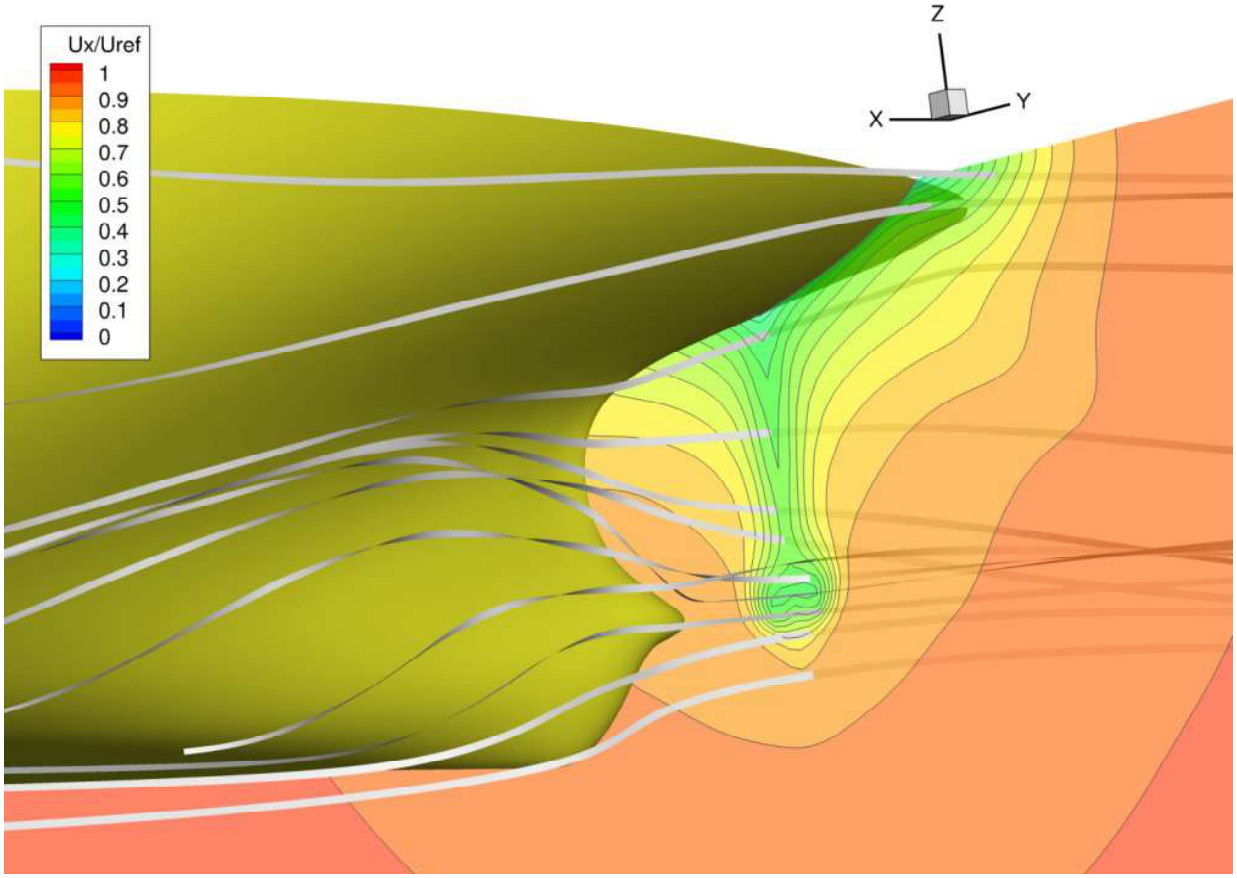
1. Introduction



Streamlines around the stern and the axial velocity field in the wake.

2. Problem

$$-\nu\nabla^2\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega$$

$$\nabla.\mathbf{u} = 0 \quad \text{in} \quad \Omega.$$

$\mathbf{u}$ is the fluid velocity vector

p is the pressure field

$\nu > 0$ is the kinematic viscosity coefficient ($1/Re$).

$\Omega \subset \mathbf{R}^{2 \text{ or } 3}$ is a bounded domain with the boundary condition:

$$\mathbf{u} = \mathbf{w} \ \text{ on } \ \partial\Omega_D, \quad \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \ \text{ on } \ \partial\Omega_N.$$

Linear system

Matrix form after linearization and discretization:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

where $F \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$, $f \in \mathbb{R}^n$ and $m \leq n$

- $F = \nu A$ in Stokes problem, A is vector Laplacian matrix
- $F = \nu A + N$ in Picard linearization, N is vector-convection matrix
- $F = \nu A + N + W$ in Newton linearization, W is the Newton derivative matrix
- B is the divergence matrix
- Sparse linear system, Symmetric indefinite (Stokes problem), nonsymmetric otherwise.
- Saddle point problem having large number of zeros on the main diagonal

3. Krylov Solvers and preconditioners

- **Direct method:**
  To solve $\mathcal{A}x = b$,
  factorize $\mathcal{A}$ into upper U and lower L triangular matrices ($LUx = b$)
  First solve $Ly = b$, then $Ux = y$

- **Classical Iterative Schemes:**
  Methods based on matrix splitting, generates sequence of iterations
  $x_{k+1} = M^{-1}(Nx_k + b) = Qx_k + s$, where $\mathcal{A} = M - N$
  Jacobi, Gauss Seidel, SOR, SSOR

- **Krylov Subspace Methods:**
  $x_{k+1} = x_k + \alpha_k p_k$
  Some well known methods are
  CGNR[1975], QMR[1991], CGS[1989], Bi-CGSTAB[1992], GMRES[1986],
  GMRESR[1994], GCR[1986], IDR(s)[2007]

IDR and IDR(s) (Induced Dimension Reduction)

- Sonneveld developed IDR in the 1970's. IDR is a finite termination (Krylov) method for solving nonsymmetric linear systems.
- Analysis showed that IDR can be viewed as Bi-CG combined with linear minimal residual steps.
- This discovery led to the development of first CGS, and later of Bi-CGSTAB (by van der Vorst).

IDR and IDR(s) (continued)

- As a result of these developments the basic IDR-idea was abandoned for the Bi-CG-approach.
- Recently, Sonneveld and van Gijzen discovered that the IDR-approach was abandoned too soon and proposed a generalization of IDR: IDR(s).
- P. SONNEVELD AND M.B. VAN GIJZEN IDR(s): a family of simple and fast algorithms for solving large nonsymmetric systems of linear equations
  SIAM J. Sci. Comput., 31, pp. 1035-1062, 2008

More information: http://ta.twi.tudelft.nl/nw/users/gijzen/IDR.html

4. ILU-type Preconditioners

A linear system $\mathcal{A}x = b$ is transformed into $P^{-1}\mathcal{A}x = P^{-1}b$ such that

- $P \approx \mathcal{A}$
- Eigenvalues of $P^{-1}\mathcal{A}$ are more clustered than $\mathcal{A}$
- $Pz = r$ cheap to compute

Several approaches, we will discuss here

- ILU preconditioner
- Preconditioned IDR(s) and Bi-CGSTAB comparison
- Block preconditioners

SILU preconditioners

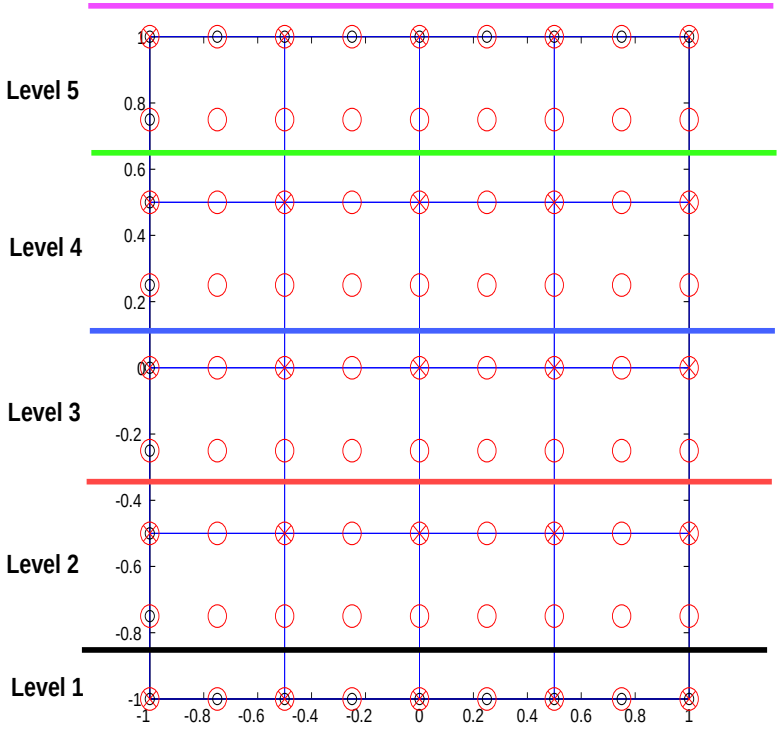
New renumbering Scheme

- Renumbering of grid points:
  - Sloan algorithm [Sloan - 1986]
  - Cuthill McKee algorithms [Cuthill McKee - 1969]
- The unknowns are reordered by p-last or p-last per level methods
  - In **p-last reordering**, first all the velocity unknowns are ordered followed by pressure unknowns. Usually it produces a large profile but avoids breakdown of LU decomposition.
  - In **p-last per level reordering**, unknowns are reordered per level such that at each level, the velocity unknowns are followed by the pressure unknowns.

what are the levels ?

SILU preconditioner

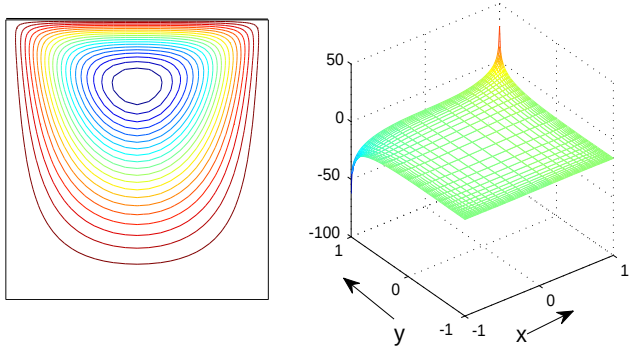
4×4 Q2-Q1 grid



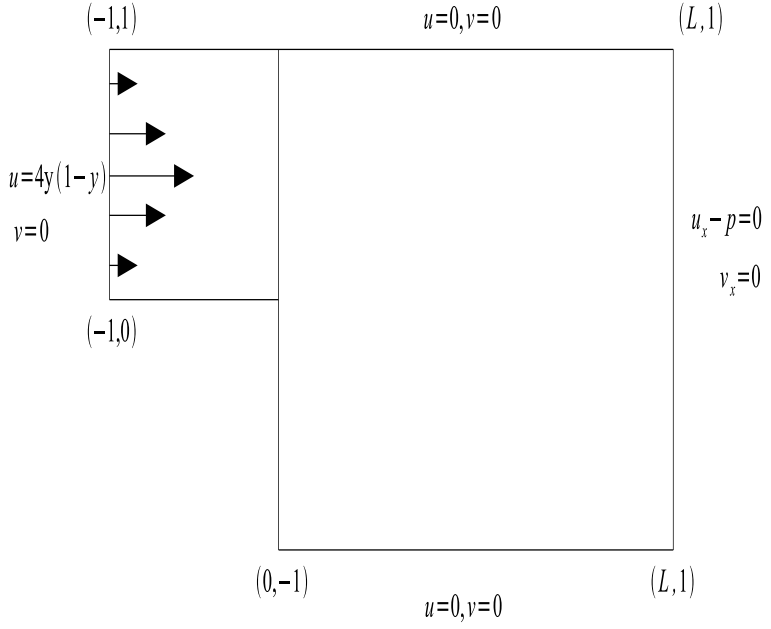
Q2-Q1 finite element subdivision

Numerical experiments (SILU preconditioner)

Driven cavity flow problem



Backward facing step problem



The iteration is stopped if the linear systems satisfy $\frac{\|r^k\|_2}{\|b\|_2} \leq tol.$

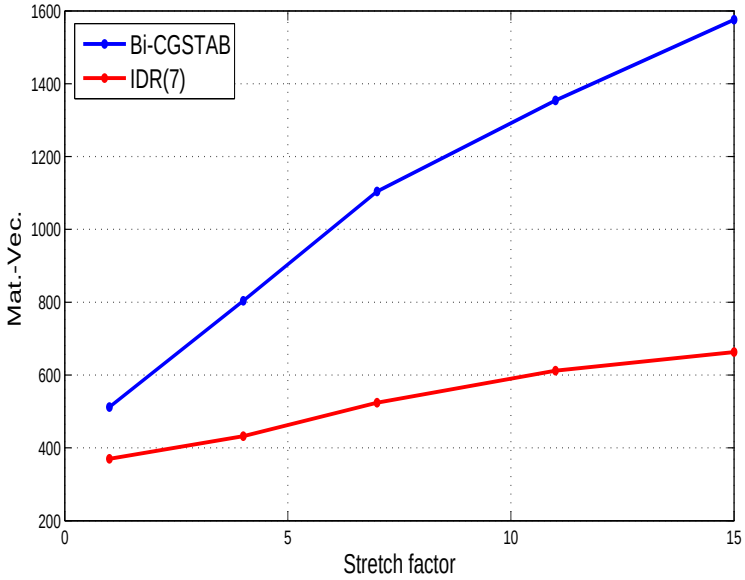
Numerical experiments (SILU preconditioners)

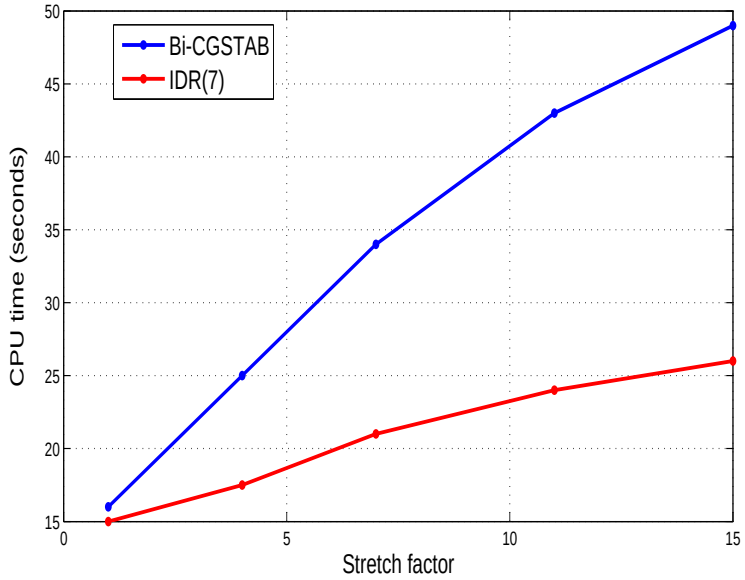
Stokes Problem in a square domain with Bi-CGSTAB, $accuracy = 10^{-6}$, Sloan renumbering

	$Q2-Q1$		$Q2-P1$	
Grid size	p-last	p-last per level	p-last	p-last per level
16×16	36(0.11)	25(0.09)	44(0.14)	34(0.13)
32×32	90(0.92)	59(0.66)	117(1.08)	75(0.80)
64×64	255(11.9)	135(6.7)	265(14)	165(9.0)
128×128	472(96)	249(52)	597(127)	407(86)

Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioned: Comparison of iterative methods for increasing stretch factor for the driven cavity Stokes problem.





Numerical Experiments (IDR(s) vs Bi-CGSTAB(l))

SILU preconditioned: Comparison of iterative methods

Driven Cavity Stokes problem, stretch factor 10

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
128×128	1104(36.5)	4	638(24.7)	6
256×256	5904(810)	6	1749(307)	8

Channel flow Stokes problem, length 100

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
64×64	1520(12)	4	938(8.7)	8
128×128	NC	6	8224(335)	8

5. Block preconditioners

$$\mathcal{A} = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BM_l^{-1} & I \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M_u^{-1} B^T \\ 0 & I \end{bmatrix}$$

$M_l = M_u = F$ and $S = -BF^{-1}B^T$ is the Schur-complement matrix.

$$\mathcal{U}_{bt} = \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}, \quad \mathcal{L}_{bt} = \mathcal{L}_b \mathcal{D}_b = \begin{bmatrix} F & 0 \\ B & \hat{S} \end{bmatrix}.$$

Preconditioners are based on combination of these blocks involve:

$Fz_1 = r_1$ The velocity subsystem

$$S \longrightarrow \hat{S}$$

$\hat{S}z_2 = r_2$ The pressure subsystem

Block preconditioners

Block triangular preconditioners

$$P_t = \mathcal{U}_{bt} = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}$$

- Pressure convection diffusion (PCD) [Kay et al, 2002]
  $\hat{S} = -A_p F_p^{-1} Q_p$, Q_p is the pressure mass matrix
- Least squares commutator (LSC) [Elman et al, 2002]
  $\hat{S} = -(BQ_u^{-1}B^T)(BQ_u^{-1}FQ_u^{-1}B^T)^{-1}(BQ_u^{-1}B^T)$, Q_u is the velocity mass matrix
- Augmented Lagrangian approach (AL) [Benzi and Olshanskii, 2006]
  F is replaced by $F_\gamma = F + \gamma B W^{-1} B^T$
  $\hat{S}^{-1} = -(\nu \hat{Q}_p^{-1} + \gamma W^{-1})$, $W = \hat{Q}_p$

Block preconditioners (SIMPLE)

SIMPLE-type preconditioners[Vuik et al-2000]

SIMPLE	SIMPLER
$z = \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1}r$	$z = \mathcal{U}_{bt}^{-1}\mathcal{L}_b^{-1}r$
	$z = z + \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1}(r - \mathcal{A}z)$
$M_u = D$	$M_l = M_u = D,\ D = diag(F)$
$\hat{S} = BD^{-1}B^T$	$\hat{S} = BD^{-1}B^T$
One Poisson solve	Two Poisson solves
One velocity solve	Two velocity solves

Lemma: In the SIMPLER preconditioner/algorithm, both variants (one or two velocity solves) are identical .

Improvements in SIMPLE-type preconditioners

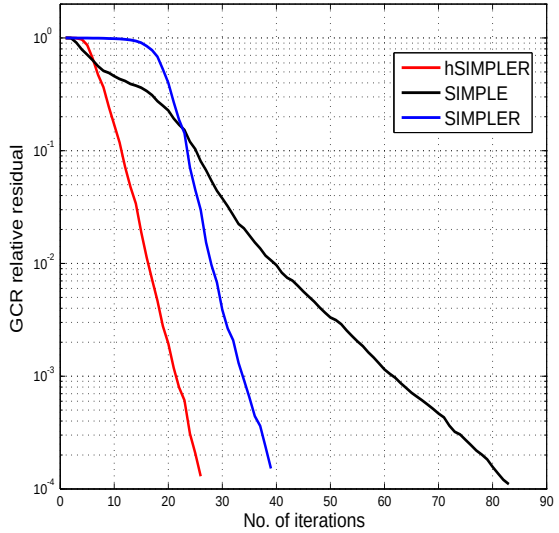
We use approximate solvers for subsystems, so flexible Krylov solvers are required (GCR, FGMRES, GMRESR)

- hSIMPLER
- MSIMPLER

Improvements in SIMPLE(R) preconditioners

hSIMPLER preconditioner:

In hSIMPLER (hybrid SIMPLER), first iteration of Krylov method preconditioned with SIMPLER is done with SIMPLE and SIMPLER is employed afterwards.



- Faster convergence than SIMPLER
- Effective in the Stokes problem

Improvements in SIMPLE(R) preconditioners

MSIMPLER preconditioner:

Making the following changes in SIMPLER leads to the MSIMPLER preconditioner.

LSC: $\hat{S} \approx -(B\hat{Q_u}^{-1}B^T)(B\hat{Q_u}^{-1}\underbrace{F\hat{Q_u}^{-1}}B^T)^{-1}(B\hat{Q_u}^{-1}B^T)$

assuming $F\hat{Q_u}^{-1} \approx I$ (time dependent problems with a small time step)

$\hat{S} = -B\hat{Q_u}^{-1}B^T$

MSIMPLER uses this approximation for the Schur complement and updates scaled with $\hat{Q_u}^{-1}$.

-Convergence better than other variants of SIMPLE

-Cheaper than SIMPLER (in construction) and LSC (per iteration)

Numerical Experiments (comparison)

3D Backward facing step: Preconditioners used in the Stokes problem with preconditioned GCR(20) with $accuracy$ of 10^{-6} (SEPRAN) using Q2-Q1 hexahedrons

Grid	SIMPLE	LSC	MSIMPLER
	iter. $(t_s)\frac{\text{in-it-}u}{\text{in-it-}p}$		
$8 \times 8 \times 16$	44(4) $\frac{97}{342}$	16(1.9) $\frac{41}{216}$	14(1.4) $\frac{28}{168}$
$16 \times 16 \times 32$	84(107) $\frac{315}{1982}$	29(51) $\frac{161}{1263}$	17(21) $\frac{52}{766}$
$24 \times 24 \times 48$	99(447) $\frac{339}{3392}$	26(233) $\frac{193}{2297}$	17(77) $\frac{46}{1116}$
$32 \times 32 \times 40$	132(972) $\frac{574}{5559}$	37(379) $\frac{233}{2887}$	20(143) $\frac{66}{1604}$

Numerical Experiments (comparison)

3D Lid driven cavity problem (tetrahedrons):The Navier-Stokes problem is solved with accuracy 10^{-4}, a linear system at each Picard step is solved with accuracy 10^{-2} using preconditioned Krylov subspace methods. Bi-CGSTAB is used as inner solver in block preconditioners(SEPRAN)

Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 16$			
20	30(20)	20(16)	144(22)
50	57(37)	37(24)	234(35)
100	120(81)	68(44)	427(62)
$32 \times 32 \times 32$			
20	38(234)	29(144)	463(353)
50	87(544)	53(300)	764(585)
100	210(1440)	104(654)	1449(1116)

Numerical Experiments (comparison)

2D Lid driven cavity problem on 64×64 stretched grid: The Stokes problem is solved with accuracy 10^{-6}. PCG is used as inner solver in block preconditioners (SEPRAN) .

Stretch factor	LSC	MSIMPLER	SILU
	GCR iter.	GCR iter.	Bi-CGSTAB iter.
1	20	17	96
8	49	28	189
16	71	34	317
32	97	45	414
64	145	56	NC
128	NC	81	NC

The Augmented Lagrangian method

$$\begin{bmatrix} F & B^T \\ B & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$ is transformed into

$$\begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} \hat{f} \\ g \end{bmatrix} \quad \text{or} \quad \mathcal{A}_{AL}\mathbf{x} = \hat{b},$$

with $\hat{f} = f + \gamma B^T W^{-1} B\, g$, where W is a non-singular matrix.

The *Ideal* AL preconditioner proposed for $\mathcal{A}_{AL}$ is

$$\mathcal{P}_{IAL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}.$$

The Augmented Lagrangian method

$$\mathcal{A}_{AL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \qquad (S_{AL} = -B(F + \gamma B^T W^{-1} B)^{-1} B^T)$$

$$\mathcal{P}_{IAL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix} \qquad (F_\gamma = F + \gamma B^T W^{-1} B)$$

- The Schur complement S_{AL} of $\mathcal{A}_{AL}$ is approximated by $-\frac{1}{\gamma} W$.
- The block F_γ becomes increasingly ill-conditioned with $\gamma \to \infty$.
- In practice it is often chosen as $\gamma = 1$, or $\gamma = O(1)$, and $W = \hat{Q}_P$.
- Open question: fast solution methods for systems with F_γ, which is denser than F and consists of mixed derivatives.

[1] M. Benzi and M.A. Olshanskii. An augmented Lagrangian-based approach to the Oseen problem. *SIAM J. Sci. Comput.*, 28:2095-2113, 2006.

The Augmented Lagrangian method

The transformed coefficient matrix $\mathcal{A}_{AL} = \begin{bmatrix} F+\gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix}$ and the ideal AL precondition $\mathcal{P}_{IAL} = \begin{bmatrix} F+\gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}$ includes (in 2D)

- the convection-diffusion block: $F = \begin{bmatrix} F_{11} & O \\ O & F_{11} \end{bmatrix}$,
- the (negative) divergence matrix: $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$,
- the modified pivot block $F_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}$.

One approximation of F_γ is $\widetilde{F}_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & O \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}$, which leads to the modified AL preconditioner $\mathcal{P}_{MAL}$ for $\mathcal{A}_{AL}$.

The Augmented Lagrangian method

$$\mathcal{P}_{IAL} = \begin{bmatrix} F_\gamma & 0 \\ B & -\frac{1}{\gamma}W \end{bmatrix} \qquad \left(F_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}\right)$$

$$\mathcal{P}_{MAL} = \begin{bmatrix} \widetilde{F}_\gamma & 0 \\ B & -\frac{1}{\gamma}W \end{bmatrix} \qquad \left(\widetilde{F}_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & 0 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}\right)$$

- systems with $\widetilde{F}_\gamma$ are easier to be solved, compared to F_γ.
- the number of iterations by using the ideal and modified AL preconditioners are both independent of the mesh refinement, and nearly independent of the Reynolds (viscosity) number.
- by using the modified AL preconditioner, there exists an optimal value of γ, which minimises the number of Krylov subspace iterations. The optimal γ is problem dependent, but mesh size independent.

Numerical experiments (Lid driven cavity)

2D lid driven cavity problem. the domain is $[0,1] \times [0,1]$. The Reynolds number is $Re = UL/\nu$, and here $U = 1$ and $L = 1$. The stretched grids are generated based on the uniform Cartesian grids with $n \times n$ cells. The stretching function is applied in both directions with parameters $a = 1/2$ and $b = 1.1$

$$x = \frac{(b+2a)c - b + 2a}{(2a+1)(1+c)}, \quad c = \left(\frac{b+1}{b-1}\right)^{\frac{\bar{x}-a}{1-a}}, \quad \bar{x} = 0, 1/n, 2/n, ..., 1.$$

Numerical experiments (Lid driven cavity)

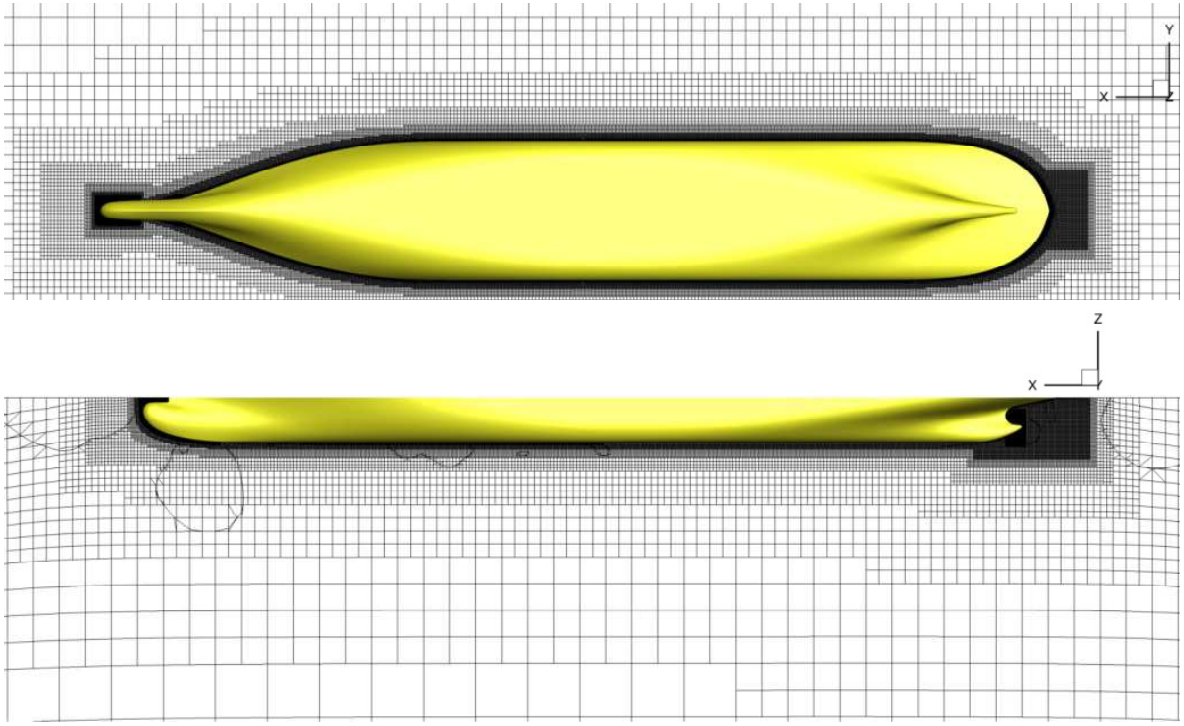
Re	100	400	1000	$2500^\star$	$5000^\star$
modified AL preconditioner					
Picard iterations:	14	27	33	66	286
GCR iterations:	5	9	11	17	19
total time:	22.7	65.1	119.6	457.7	2636.3
modified 'grad-div' preconditioner					
Picard iterations:	13	27	31	51	308
GCR iterations:	7	11	16	28	24
total time:	10.8	35.8	64.4	159.5	812.5
ideal SIMPLER preconditioner					
Picard iterations:	14	27	31	51	325
GCR iterations:	40	53	63	92	107
total time:	81.5	235.2	508.4	929.7	9548.7

Numerical experiments (Lid driven cavity)

Re	100	400	1000	2500⋆	5000⋆
modified AL preconditioner					
Newton iterations:	6	7	7	8	9
GCR iterations:	8	14	21	33	50
total time:	14.8	26.2	74.6	194.2	277.1
modified 'grad-div' preconditioner					
Newton iterations:	6	7	8	9	9
GCR iterations:	10	17	28	53	77
total time:	8.5	15.7	32.7	119.1	167.9
modified SIMPLER preconditioner					
Newton iterations:	10	8⋆	8⋆	11	15
GCR iterations:	43	82	84	80	90
total time:	68.3	102.9	232.8	203.2	561.6

6. Maritime Applications

Container vessel (unstructured grid)



RaNS equations

k-ω turbulence model

$y^+ \approx 1$

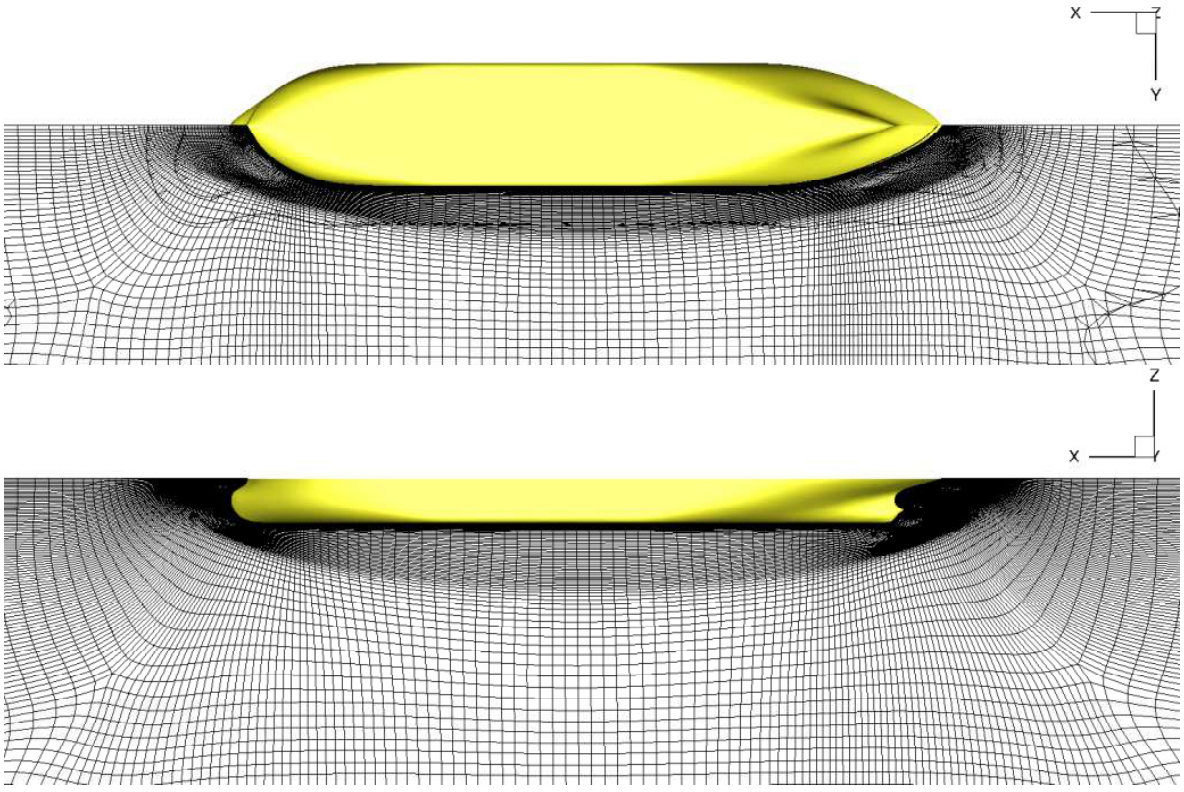
Model-scale:

$\mathrm{Re} = 1.3 \cdot 10^7$

13.3m cells

max aspect ratio $1 : 1600$

Tanker (block-structured grid)



Model-scale:

$\mathrm{Re} = 4.6 \cdot 10^6$

2.0m cells

max aspect ratio $1 : 7000$

Full-scale:

$\mathrm{Re} = 2.0 \cdot 10^9$

2.7m cells

max aspect ratio $1 : 930\,000$

Discretization

Co-located, cell-centered finite volume discretization of the steady Navier-Stokes equations with Picard linearization leads to linear system:

$$\begin{bmatrix} Q_1 & 0 & 0 & G_1 \\ 0 & Q_2 & 0 & G_2 \\ 0 & 0 & Q_3 & G_3 \\ D_1 & D_2 & D_3 & C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ g \end{bmatrix} \qquad \text{for brevity:} \quad \begin{bmatrix} Q & G \\ D & C \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

with $Q_1 = Q_2 = Q_3$.

$\Rightarrow$ Solve system with FGMRES and SIMPLE-type preconditioner
Turbulence equations (k-ω model) remain segregated

SIMPLE-method

Given u^k and p^k:

1. solve $Qu^* = f - Gp^k$
2. solve $(C - DQ^{-1}G)p' = g - Du^* - Cp^k$
3. compute $u' = -Q^{-1}Gp'$
4. update $u^{k+1} = u^* + u'$ and $p^{k+1} = p^k + p'$

with the SIMPLE approximation $Q^{-1} \approx \mathrm{diag}(Q)^{-1}$.
$\Rightarrow$ "Matrix-free": only assembly and storage of Q and $(C - DQ^{-1}G)$. For D, G and C the action suffices.

SIMPLER: additional pressure prediction

Given u^k and p^k, start with a pressure prediction:

1. solve $(C - D\,\mathrm{diag}(Q)^{-1}G)p^* = g - Du^k - D\,\mathrm{diag}(Q)^{-1}(f - Qu^k)$
2. continue with SIMPLE using p^* instead of p^k

Some practical constraints

Compact stencils are preferred on unstructured grids:

- neighbors of cell readily available; neighbors of neighbors not

Also preferred because of MPI parallel computation:

- domain decomposition, communication

Compact stencil?

- ✓ Matrix $Q_1 (= Q_2 = Q_3)$, thanks to defect correction
- ✗ Stabilization matrix C

$\Rightarrow$ modify SIMPLE(R) such that C is not required on the l.h.s.

Treatment of stabilization matrix

- In SIMPLE, neglect C in l.h.s. of pressure correction equation

$$(C - D\mathrm{diag}(Q)^{-1}G)p' = g - Du^* - Cp^k$$
$$\Downarrow$$
$$-D\mathrm{diag}(Q)^{-1}Gp' = g - Du^* - Cp^k$$

- In SIMPLER, do *not* involve the mass equation when deriving the pressure prediction p^*

$$(C - D\,\mathrm{diag}(Q)^{-1}G)p^* = g - Du^k - D\,\mathrm{diag}(Q)^{-1}(f - Qu^k)$$
$$\Downarrow$$
$$-D\,\mathrm{diag}(Q)^{-1}Gp^* = -D\,\mathrm{diag}(Q)^{-1}(f - Qu^k)$$

Container vessel

Tables show number of non-linear iterations and wall clock time needed to converge to machine precision, starting from uniform flow.

Model-scale $\mathrm{Re} = 1.3 \cdot 10^7$, max cell aspect ratio $1 : 1600$

grid	CPU cores	SIMPLE		KRYLOV-SIMPLER	
		# its	Wall clock	# its	Wall clock
13.3m	128	3187	5h 26mn	427	3h 27mn

Tanker

Model-scale $\mathrm{Re} = 4.6 \cdot 10^6$, max cell aspect ratio $1 : 7000$

grid	CPU cores	SIMPLE		KRYLOV-SIMPLER	
		its	Wall clock	its	Wall clock
0.25m	8	1379	25mn	316	29mn
0.5m	16	1690	37mn	271	25mn
1m	32	2442	57mn	303	35mn
2m	64	3534	1h 29mn	519	51mn

Full-scale $\mathrm{Re} = 2.0 \cdot 10^9$, max cell aspect ratio $1 : 930\,000$

grid	CPU cores	SIMPLE		KRYLOV-SIMPLER	
		its	Wall clock	its	Wall clock
2.7m	64	29 578	16h 37mn	1330	3h 05mn

7. Conclusions

- *MSIMPLER is at present the fastest of all SIMPLE-type preconditioners.*
- *In our experiments, MSIMPLER proved to be cheaper than SILU, especially when the problem is solved with high accuracy.*
- *MSIMPLER shows better performance than LSC. Both have similar convergence characteristics.*
- *For academic problems, Modified Augmented Lagrangian (MAL) and grad-div are nearly independent of the grid size and Reynolds number*
- *MAL/grad-div are faster than (M)SIMPLER*
- *Future research: MAL/grad-div for industrial (Maritime) applications*

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