# Coupled preconditioners for the Incompressible Navier Stokes Equations

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# Messages

1. Incompressible Navier-Stokes are important
2. Much progress in solvers for academic test problems
3. Transfer methods to industrial problems

# Outline

1. Introduction
2. Problem
3. Krylov solvers
4. Block preconditioners
   - SIMPLE
   - Augmented Lagrangian
5. Maritime Applications
6. Conclusions

# 1. Introduction

## Flooding of the Netherlands, 1953

## 2. Problem

$$-\nu\nabla^2\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega$$

$$\nabla.\mathbf{u} = 0 \quad \text{in} \quad \Omega.$$

$\mathbf{u}$ is the fluid velocity vector

$p$ is the pressure field

$\nu > 0$ is the kinematic viscosity coefficient ( $1/Re$).

$\Omega \subset \mathbf{R}^{2 \text{ or } 3}$ is a bounded domain with the boundary condition:

$$\mathbf{u} = \mathbf{w} \;\text{ on }\; \partial\Omega_D, \quad \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \;\text{ on }\; \partial\Omega_N.$$

# Linear system

Matrix form after linearization and discretization:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

where $F \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $f \in \mathbb{R}^n$ and $m \leq n$

- $F = \nu A$ in Stokes problem, $A$ is vector Laplacian matrix
- $F = \nu A + N$ in Picard linearization, $N$ is vector-convection matrix
- $F = \nu A + N + W$ in Newton linearization, $W$ is the Newton derivative matrix
- $B$ is the divergence matrix
- Sparse linear system, Symmetric indefinite (Stokes problem), nonsymmetric otherwise.
- Saddle point problem having large number of zeros on the main diagonal

# 3. Krylov Solvers and preconditioners

- **Direct method:**
  To solve $\mathcal{A}x = b$,
  factorize $\mathcal{A}$ into upper $U$ and lower $L$ triangular matrices ($LUx = b$)
  First solve $Ly = b$, then $Ux = y$

- **Classical Iterative Schemes:**
  Methods based on matrix splitting, generates sequence of iterations
  $x_{k+1} = M^{-1}(Nx_k + b) = Qx_k + s$, where $\mathcal{A} = M - N$
  Jacobi, Gauss Seidel, SOR, SSOR

- **Krylov Subspace Methods:**
  $x_{k+1} = x_k + \alpha_k p_k$
  Some well known methods are
  CGNR[1975], QMR[1991], CGS[1989], Bi-CGSTAB[1992], GMRES[1986], FGMRES[1992], GMRESR[1994], GCR[1986], IDR($s$)[2007]

# 4. Block preconditioners

$$\mathcal{A} = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BM_l^{-1} & I \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M_u^{-1}B^T \\ 0 & I \end{bmatrix}$$

$M_l = M_u = F$ and $S = -BF^{-1}B^T$ is the Schur-complement matrix.

$$\mathcal{U}_{bt} = \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}, \quad \mathcal{L}_{bt} = \mathcal{L}_b \mathcal{D}_b = \begin{bmatrix} F & 0 \\ B & \hat{S} \end{bmatrix}.$$

Preconditioners based on a combination of these blocks involve the following subsystems:

$Fz_1 = r_1$ The velocity subsystem

$$S \longrightarrow \hat{S}$$

$\hat{S}z_2 = r_2$ The pressure subsystem

# Block preconditioners

**Block triangular preconditioners**

$$P_t = \mathcal{U}_{bt} = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}$$

- Pressure convection diffusion (PCD) [Kay et al, 2002]
  $\hat{S} = -A_p F_p^{-1} Q_p$, $Q_p$ is the pressure mass matrix
- Least squares commutator (LSC) [Elman et al, 2002]
  $\hat{S} = -(BQ_u^{-1}B^T)(BQ_u^{-1}FQ_u^{-1}B^T)^{-1}(BQ_u^{-1}B^T)$, $Q_u$ is the velocity mass matrix
- Augmented Lagrangian approach (AL) [Benzi and Olshanskii, 2006]
  $F$ is replaced by $F_\gamma = F + \gamma B W^{-1} B^T$
  $\hat{S}^{-1} = -(\nu \hat{Q}_p^{-1} + \gamma W^{-1}),\ W = \hat{Q}_p$





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# Block preconditioners (SIMPLE)

**SIMPLE-type preconditioners**[Vuik et al-2000]

| SIMPLE | SIMPLER |
|---|---|
| $z = \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1} r$ | $z = \mathcal{U}_{bt}^{-1}\mathcal{L}_b^{-1} r$ |
| | $z = z + \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1}(r - \mathcal{A}z)$ |
| $M_u = D$ | $M_l = M_u = D$, $D = diag(F)$ |
| $\hat{S} = BD^{-1}B^T$ | $\hat{S} = BD^{-1}B^T$ |
| One Poisson solve | Two Poisson solves |
| One velocity solve | Two velocity solves |

**Lemma:** In the SIMPLER preconditioner/algorithm, both variants (one or two velocity solves) are identical .

# Improvements in SIMPLE(R) preconditioners

## MSIMPLER preconditioner:

Making the following changes in SIMPLER leads to the MSIMPLER preconditioner.
LSC: $\hat{S} \approx -(B\hat{Q_u}^{-1}B^T)(B\hat{Q_u}^{-1}\underbrace{F\hat{Q_u}^{-1}}B^T)^{-1}(B\hat{Q_u}^{-1}B^T)$

assuming $F\hat{Q_u}^{-1} \approx I$ (time dependent problems with a small time step)

$\hat{S} = -B\hat{Q_u}^{-1}B^T$

MSIMPLER uses this approximation for the Schur complement and updates scaled with $\hat{Q_u}^{-1}$.

-Convergence better than other variants of SIMPLE
-Cheaper than SIMPLER (in construction) and LSC (per iteration)

# Numerical Experiments (comparison)

3D Backward facing step: Preconditioners used in the Stokes problem with preconditioned GCR(20) with $accuracy$ of $10^{-6}$ (SEPRAN) using Q2-Q1 hexahedrons

| Grid | SIMPLE | LSC | MSIMPLER |
|---|---|---|---|
| iter. $(t_s)\,\frac{\text{in-it-}u}{\text{in-it-}p}$ | | | |
| $8 \times 8 \times 16$ | 44(4) $\frac{97}{342}$ | 16(1.9) $\frac{41}{216}$ | 14(1.4) $\frac{28}{168}$ |
| $16 \times 16 \times 32$ | 84(107) $\frac{315}{1982}$ | 29(51) $\frac{161}{1263}$ | 17(21) $\frac{52}{766}$ |
| $24 \times 24 \times 48$ | 99(447) $\frac{339}{3392}$ | 26(233) $\frac{193}{2297}$ | 17(77) $\frac{46}{1116}$ |
| $32 \times 32 \times 40$ | 132(972) $\frac{574}{5559}$ | 37(379) $\frac{233}{2887}$ | 20(143) $\frac{66}{1604}$ |

# Numerical Experiments (comparison)

2D Lid driven cavity problem on $64 \times 64$ stretched grid: The Stokes problem is solved with accuracy $10^{-6}$. PCG is used as inner solver in block preconditioners (SEPRAN) .

| Stretch factor | LSC | MSIMPLER | SILU |
|---|---|---|---|
| | GCR iter. | GCR iter. | Bi-CGSTAB iter. |
| 1 | 20 | 17 | 96 |
| 8 | 49 | 28 | 189 |
| 16 | 71 | 34 | 317 |
| 32 | 97 | 45 | 414 |
| 64 | 145 | 56 | NC |
| 128 | NC | 81 | NC |

# Augmented Lagrangian [Benzi and Olshanskii, 2006]

$$\begin{bmatrix} F & B^T \\ B & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$ transformed to

$$\begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} \hat{f} \\ g \end{bmatrix} \quad \text{or} \quad \mathcal{A}_{AL}\mathbf{x} = \hat{b},$$

$\hat{f} = f + \gamma B^T W^{-1} B \, g$, and $W$ is non-singular.

*Ideal* AL preconditioner for $\mathcal{A}_{AL}$ is

$$\mathcal{P}_{IAL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}.$$

# The Augmented Lagrangian method

$$\mathcal{A}_{AL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \qquad (S_{AL} = -B(F + \gamma B^T W^{-1} B)^{-1} B^T)$$

$$\mathcal{P}_{IAL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix} \qquad (F_\gamma = F + \gamma B^T W^{-1} B)$$

- $S_{AL}$ of $\mathcal{A}_{AL}$ is approximated by $-\frac{1}{\gamma} W$.
- $F_\gamma$ becomes ill-conditioned for $\gamma \to \infty$.
- In practice $\gamma = 1$, or $\gamma = O(1)$, and $W = \hat{Q}_P$.
- Open question: fast solution methods for systems with $F_\gamma$, which is denser than $F$ and consists of mixed derivatives.

[1] M. Benzi and M.A. Olshanskii. An augmented Lagrangian-based approach to the Oseen problem. *SIAM J. Sci. Comput.*, 28:2095-2113, 2006.

# The Augmented Lagrangian method

$\mathcal{A}_{AL} = \begin{bmatrix} F+\gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix}$ and the ideal AL precondition

$\mathcal{P}_{IAL} = \begin{bmatrix} F+\gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}$ includes (in 2D)

- the convection-diffusion block: $F = \begin{bmatrix} F_{11} & O \\ O & F_{11} \end{bmatrix}$,
- the (negative) divergence matrix: $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$,
- the modified pivot block $F_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}$.

One approximation of $F_\gamma$ is $\widetilde{F}_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & O \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}$, which leads to the modified AL preconditioner $\mathcal{P}_{MAL}$ for $\mathcal{A}_{AL}$.

# The Augmented Lagrangian method (summary)

$$\mathcal{P}_{IAL} = \begin{bmatrix} F_\gamma & 0 \\ B & -\frac{1}{\gamma}W \end{bmatrix} \qquad \left(F_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}\right)$$

$$\mathcal{P}_{MAL} = \begin{bmatrix} \widetilde{F}_\gamma & 0 \\ B & -\frac{1}{\gamma}W \end{bmatrix} \qquad \left(\widetilde{F}_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & 0 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}\right)$$

- systems with $\widetilde{F}_\gamma$ are easier to be solved, compared to $F_\gamma$.
- the number of iterations by using the ideal and modified AL preconditioners are both independent of the mesh refinement, and nearly independent of the Reynolds (viscosity) number.
- by using the modified AL preconditioner, there exists an optimal value of $\gamma$, which minimises the number of Krylov subspace iterations. The optimal $\gamma$ is problem dependent, but mesh size independent.

# Numerical experiments (Lid driven cavity)

2D lid driven cavity problem. the domain is $[0,1]\times[0,1]$. The Reynolds number is $Re = UL/\nu$, and here $U = 1$ and $L = 1$. The stretched grids are generated based on the uniform Cartesian grids with $n \times n$ cells. The stretching function is applied in both directions with parameters $a = 1/2$ and $b = 1.1$

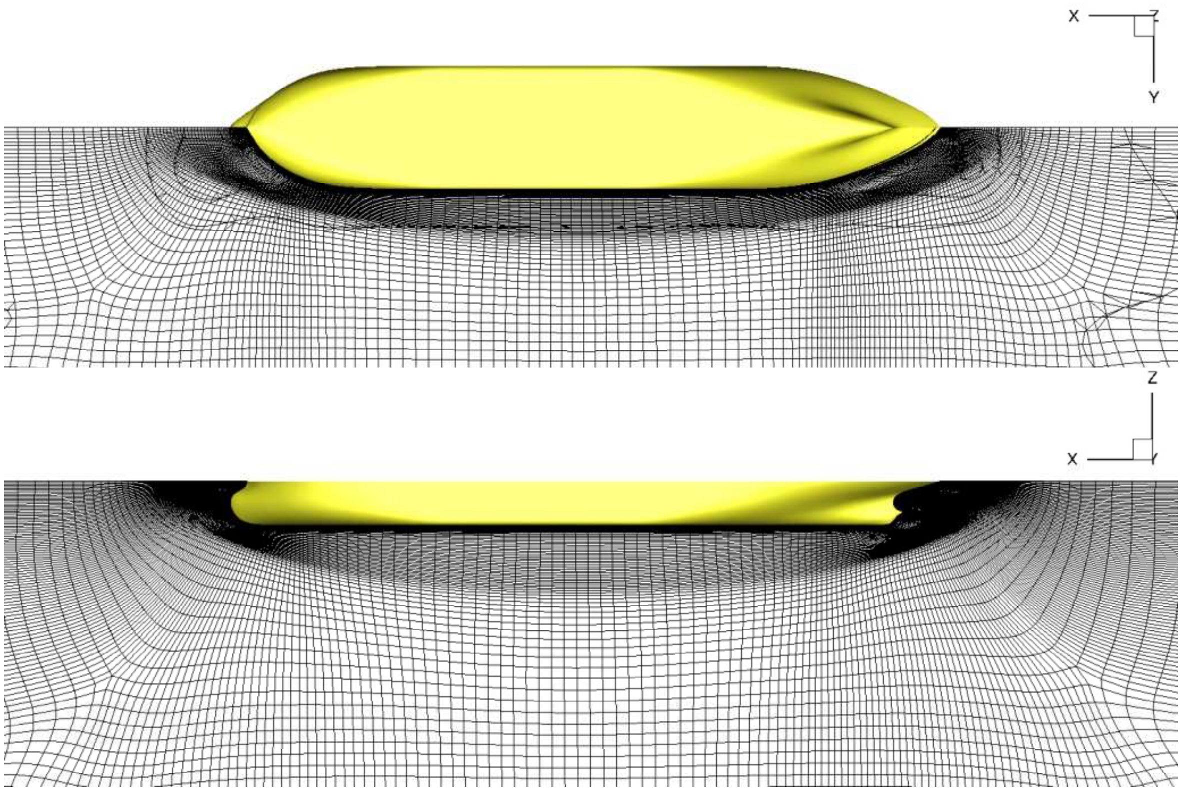
$$x = \frac{(b+2a)c - b + 2a}{(2a+1)(1+c)}, \quad c = (\frac{b+1}{b-1})^{\frac{\bar{x}-a}{1-a}}, \quad \bar{x} = 0, 1/n, 2/n, ..., 1.$$

# Numerical experiments (Lid driven cavity)

| Re | 100 | 400 | 1000 | $2500^\star$ | $5000^\star$ |
|---|---|---|---|---|---|
| modified AL preconditioner | | | | | |
| Newton iterations: | 6 | 7 | 7 | 8 | 9 |
| GCR iterations: | 8 | 14 | 21 | 33 | 50 |
| total time: | 14.8 | 26.2 | 74.6 | 194.2 | 277.1 |
| modified 'grad-div' preconditioner | | | | | |
| Newton iterations: | 6 | 7 | 8 | 9 | 9 |
| GCR iterations: | 10 | 17 | 28 | 53 | 77 |
| total time: | 8.5 | 15.7 | 32.7 | 119.1 | 167.9 |
| modified SIMPLER preconditioner | | | | | |
| Newton iterations: | 10 | $8^\star$ | $8^\star$ | 11 | 15 |
| GCR iterations: | 43 | 82 | 84 | 80 | 90 |
| total time: | 68.3 | 102.9 | 232.8 | 203.2 | 561.6 |

# 5. Maritime Applications

Tanker (block-structured grid)




Model-scale:

$\mathrm{Re} = 4.6 \cdot 10^6$

2.0m cells

max aspect ratio $1 : 7000$

Full-scale:

$\mathrm{Re} = 2.0 \cdot 10^9$

2.7m cells

max aspect ratio $1 : 930\,000$





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# Discretization

Co-located, cell-centered finite volume discretization of the steady Navier-Stokes equations with Picard linearization leads to linear system:

$$\begin{bmatrix} Q_1 & 0 & 0 & G_1 \\ 0 & Q_2 & 0 & G_2 \\ 0 & 0 & Q_3 & G_3 \\ D_1 & D_2 & D_3 & C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ g \end{bmatrix}$$

for brevity: $\begin{bmatrix} Q & G \\ D & C \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$

with $Q_1 = Q_2 = Q_3$.

$\Rightarrow$ Solve system with FGMRES and SIMPLE-type preconditioner
Turbulence equations ($k$-$\omega$ model) remain segregated

# Tanker

Model-scale $\mathrm{Re} = 4.6 \cdot 10^6$, max cell aspect ratio $1 : 7000$

| grid | CPU cores | SIMPLE | | KRYLOV-SIMPLER | |
|---|---|---|---|---|---|
| | | its | Wall clock | its | Wall clock |
| 0.25m | 8 | 1379 | 25mn | 316 | 29mn |
| 0.5m | 16 | 1690 | 37mn | 271 | 25mn |
| 1m | 32 | 2442 | 57mn | 303 | 35mn |
| 2m | 64 | 3534 | 1h 29mn | 519 | 51mn |

Full-scale $\mathrm{Re} = 2.0 \cdot 10^9$, max cell aspect ratio $1 : 930\,000$

| grid | CPU cores | SIMPLE | | KRYLOV-SIMPLER | |
|---|---|---|---|---|---|
| | | its | Wall clock | its | Wall clock |
| 2.7m | 64 | 29 578 | 16h 37mn | 1330 | 3h 05mn |

# Augmented Lagrangian for finite volumes

Stabilised coupled velocity-pressure system:

$$\mathcal{A} = \begin{bmatrix} Q & G \\ D & C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix},$$

Stabilization matrix $C$, is given by

$$C = D\text{diag}(Q)^{-1}G + \text{diag}(Q)^{-1}L,$$

where $L$ is the Laplacian matrix.

# Preconditioners used

$$\mathcal{P}_{USER} = \begin{bmatrix} Q & G \\ O & \widetilde{S} \end{bmatrix}$$

$$\mathcal{A}_{AL1} = \begin{bmatrix} Q_\gamma & G_\gamma \\ D & C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f}_\gamma \\ g \end{bmatrix}, \mathcal{P}_{MAL1} = \begin{bmatrix} \widetilde{Q}_\gamma & G_\gamma \\ O & C + \frac{1}{\gamma} W \end{bmatrix},$$

with $Q_\gamma = Q - \gamma G W^{-1} D$, $G_\gamma = G - \gamma G W^{-1} C$ and $\mathbf{f}_\gamma = \mathbf{f} - \gamma G W^{-1} g$.

$$\mathcal{A}_{AL2} = \begin{bmatrix} Q_\gamma & G_\gamma \\ D_\gamma & C_\gamma \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f}_\gamma \\ g_\gamma \end{bmatrix}, \mathcal{P}_{MAL2} = \begin{bmatrix} \widetilde{Q}_\gamma & G_\gamma \\ O & C_\gamma + \frac{1}{\gamma} W \end{bmatrix}$$

with $D_\gamma = D + \gamma C W^{-1} D$, $C_\gamma = C + \gamma C W^{-1} C$ and $g_\gamma = g + \gamma C W^{-1} g$

# Numerical experiments

Flate plate (academic test problem) in ReFRESCO

PETSc solver

- For the velocity-pressure coupled system: FGMRES with tolerance $0.001$.
- For the 3 sub-momentum systems in $\widetilde{Q}_{\gamma}$: GMRES+BJACOBI with tolerance $0.01$.
- For the 1 sub-system with the approximation of Schur: GMRES+BJACOBI with tolerance $0.01$.

For the nonlinear iterations, tolerance is $10^{-10}$.

# MAL1 preconditioner (choice of $\gamma$)

| Grid | $32^2$ | $44^2$ | $64^2$ | $88^2$ | $128^2$ |
|---|---|---|---|---|---|
| $\gamma = 0.3$ | | | | | |
| nonlinear iter. | 112 | 96 | 90 | 88 | 85 |
| Linear iter. | 25 | 30 | 35 | 42 | 49 |
| $\gamma = 0.4$ | | | | | |
| nonlinear iter. | 125 | 121 | 116 | 112 | 117 |
| Linear iter. | 16 | 18 | 20 | 22 | 23 |
| $\gamma = 0.5$ | | | | | |
| nonlinear iter. | 300 | 300 | 281 | 279 | 270 |
| Linear iter. | 7 | 9 | 10 | 10 | 10 |
| $\gamma = 0.8$ | | | | | |
| nonlinear iter. | $> 300$ | $> 300$ | $> 300$ | $> 300$ | $> 300$ |
| Linear iter. | 4 | 5 | 5 | 5 | 6 |

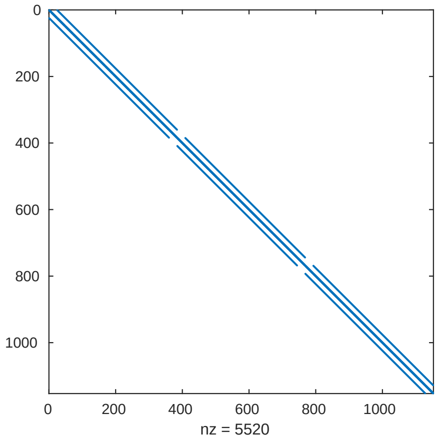
# Comparison of preconditioners

| Grid | $32^2$ | $44^2$ | $64^2$ | $88^2$ | $128^2$ |
|---|---|---|---|---|---|
| $\mathcal{P}_{MAL1}$ for $\mathcal{A}_{AL1}$ with $\gamma = 0.4$ | | | | | |
| nonlinear iter. | 125 | 121 | 116 | 112 | 117 |
| Linear iter. | 16 | 18 | 20 | 22 | 23 |
| $\mathcal{P}_{MAL2}$ for $\mathcal{A}_{AL2}$ with $\gamma = 0.4$ | | | | | |
| nonlinear iter. | 118 | 113 | 107 | 108 | 106 |
| Linear iter. | 18 | 19 | 21 | 22 | 24 |
| $\mathcal{P}_{USER}$ for $\mathcal{A}$ | | | | | |
| nonlinear iter. | 123 | 99 | 110 | 95 | 92 |
| Linear iter. | 20 | 25 | 30 | 50 | 80 |

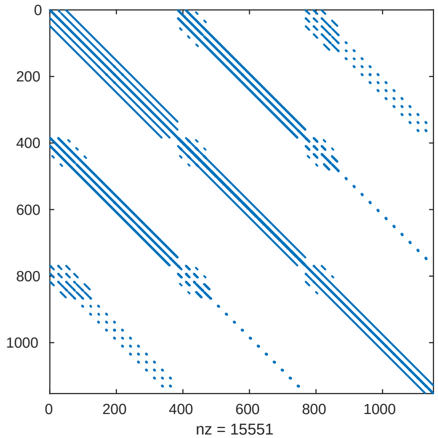
# Number of sub-systems iterations

| Grid | $32^2$ | $44^2$ | $64^2$ | $88^2$ | $128^2$ |
|---|---|---|---|---|---|
| $\mathcal{P}_{MAL1}$ for $\mathcal{A}_{AL1}$ with $\gamma = 0.4$ | | | | | |
| velocity sub-system iter. | 6 | 9 | 12 | 15 | 20 |
| pressure sub-system iter. | 2 | 2 | 2 | 2 | 2 |
| $\mathcal{P}_{MAL2}$ for $\mathcal{A}_{AL2}$ with $\gamma = 0.4$ | | | | | |
| velocity sub-system iter. | 6 | 9 | 12 | 15 | 20 |
| pressure sub-system iter. | 2 | 2 | 2 | 2 | 3 |

# Sparsity of the blocks

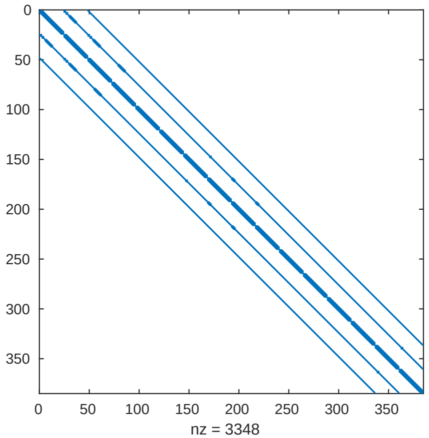



(a) $Q$

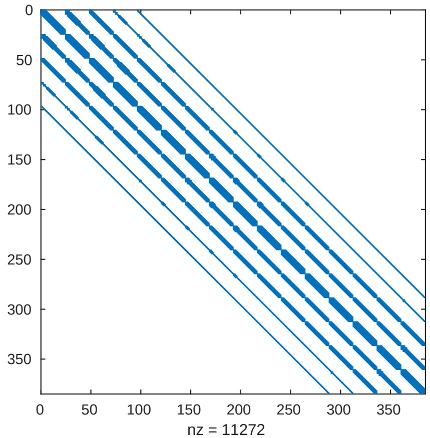



(b) $Q_\gamma = Q - \gamma G W^{-1} D$

# Sparsity of the blocks




(c) $C$(SPD)




(d) $C + \gamma C W^{-1} C$(SPD)

# 6. Conclusions

- *MSIMPLER is at present the fastest of all SIMPLE-type preconditioners.*
- *MSIMPLER shows better performance than LSC. Both have similar convergence characteristics.*
- *For academic problems (FEM), Modified Augmented Lagrangian (MAL) and grad-div are nearly independent of the grid size and Reynolds number*
- *MAL/grad-div are faster than (M)SIMPLER*
- *Future research: MAL/grad-div for industrial (Maritime) applications (FVM)*

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