

Deflating the Shifted Laplacian for the Helmholtz Equation: Fourier Analysis and Numerical Results

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Sketch of the Talk

- Helmholtz and Shifted Laplacian
- Deflation preconditioning
- Variations in Deflation for Helmholtz
- Analysis/Comparison
- Results
- Conclusions

The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x_i) - k^2(x_i) \mathbf{u}(x_i) = \mathbf{g}(x_i) \quad \text{in } \Omega$$

- Linear system $A_h u_h = g_h$ is:
 - Sparse & complex valued, for certain boundary conditions
 - Symmetric & Indefinite for large k
- For high resolution a very fine grid is required: 10 – 20 gridpoints per wavelength $\rightarrow A_h$ is extremely large!
- Standard multigrid method does not work!
- Traditionally solved by a Krylov subspace method, which exploits the **sparsity**.

Complex Shifted Laplace Preconditioner

$$M(\beta_1, \beta_2) := -\Delta - (\beta_1 - \iota\beta_2)k^2 I$$

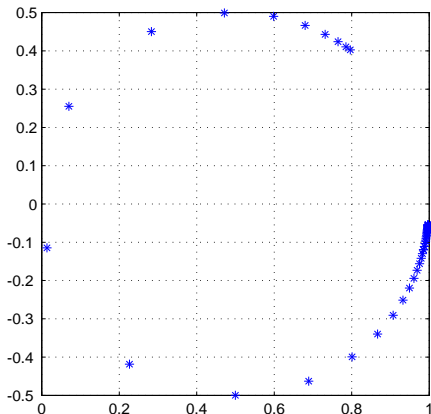
Advantage: Spectrum is bounded in circle.

Disadvantage : That circle touches the origin 0;

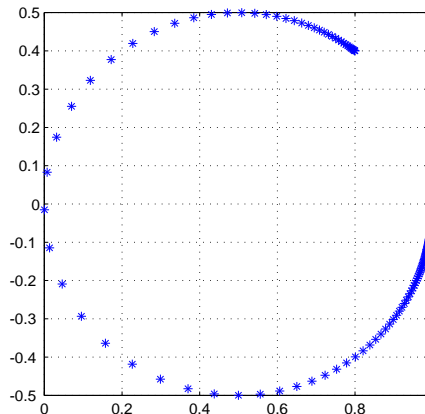
Spectrum contains near-zero eigenvalues for large k .

Spectrum of CSLP preconditioned Helmholtz

$k = 30$



$k = 120$



Deflation

Deflation, a projection preconditioner

$$P = I - AQ, \quad \text{with } Q = ZE^{-1}Z^T \quad \text{and } E = Z^T AZ$$

where,

$$Z \in R^{n \times r}, \quad \text{with deflation vectors } Z = [z_1, \dots, z_r], \quad \text{rank}(Z) = r \leq n$$

Along with a traditional preconditioner M , deflated preconditioned system reads

$$PM^{-1}Au = PM^{-1}g.$$

The choice of deflation vectors: spectrum of matrix, physics of problem, etc

Deflation for Helmholtz

With choice of multigrid inter-grid transfer operator (Prolongation) as deflation matrix, i.e. $Z = I_{2h}^h$ and $Z^T = I_h^{2h}$ then

$$P_h = I_h - A_h Q_h, \quad \text{with} \quad Q_h = I_{2h}^h A_{2h}^{-1} I_h^{2h} \quad \text{and} \quad A_{2h} = I_h^{2h} A_h I_{2h}^h$$

where

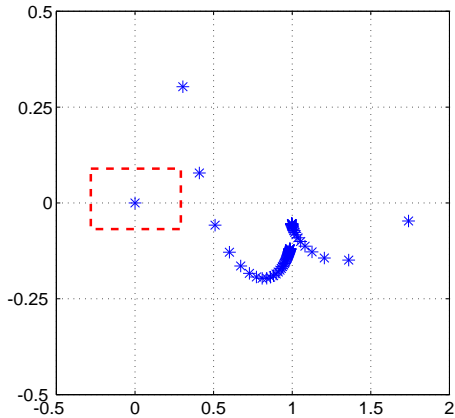
P_h can be interpreted as a coarse grid correction and

Q_h as the coarse grid operator

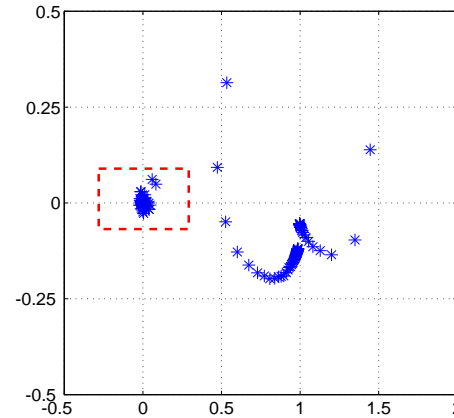
A_{2h}^{-1} How to solve this ? ?

MultiLevel approach; Krylov approximation of A_{2h}^{-1} preconditioned by CSLP and deflation again.

Deflation: Approximate solve A_{2h}^{-1}



Exact inversion of A_{2h}



Inexact inversion of A_{2h}

Shifting Deflated-Spectrum

Shift term

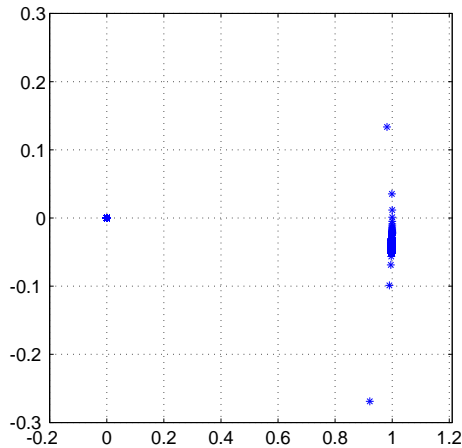
$$Q_h = I_h^{2h} A_{2h}^{-1} I_h^{2h}$$

Strategy: Solve A_{2h} iteratively to required accuracy on certain levels, and shift the deflated spectrum to λ_h^n by adding shift in deflation preconditioner, call it **ADEF1** preconditioner

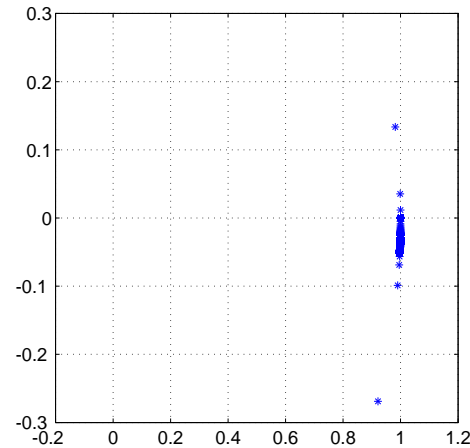
$$P_{(h,ADEF1)} = M_h^{-1} P_h + \lambda_h^n Q_h$$

It is theoretically shown that term Q_h shifts the spectrum to λ_h^n

Deflation: Shift to 1 ?



Without Shift Q_{2h}



With Shift Q_{2h}

NEXT: $\lambda_h(B_{h,2h})$ where $B_{h,2h} = P_{(h,ADEF1)} M_h^{-1} A_h$

Spectral formula

If $\mathbf{c}_\ell = \cos(l\pi h)$, spectral formulae of $P_{h,ADEF}A_h$ is

$$\lambda_h (P_{h,ADEF}A_h) = -\frac{(\mathbf{c}_\ell^2 + 1) \kappa^4 + (-4 \mathbf{c}_\ell^2 - 4) \kappa^2 - 4 (\mathbf{c}_\ell^4 - 1)}{((\mathbf{c}_\ell^2 + 1)\kappa^2 + 2(\mathbf{c}_\ell^2 - 1)) h^2}$$

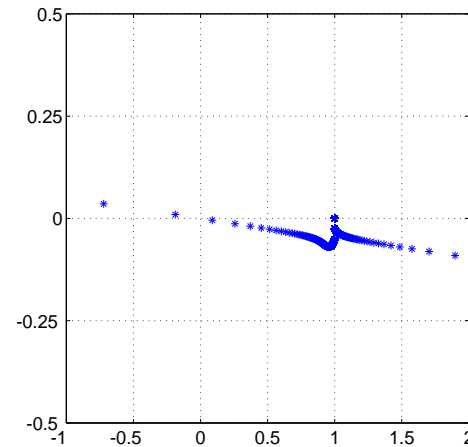
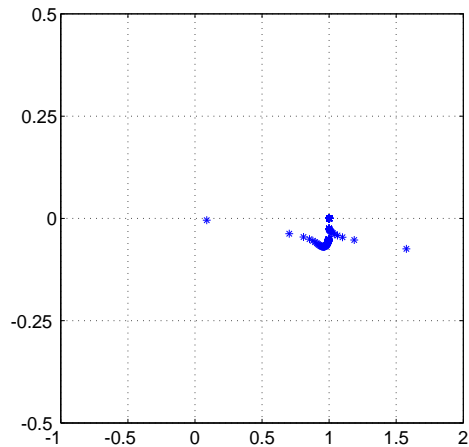
We also know, eigenvalues of Galerikin Helmholtz operator

$$A_{2h} = (I_h^{2h})^\ell A_h^\ell (I_{2h}^h)^\ell = \frac{2(1 - c_\ell^2) - \kappa^2(1 + c_\ell^2)}{2h^2}$$

Denominator in $\lambda_h(P_{h,ADEF1}A_h)$ is scaled formula of A_{2h}

Spectrum insights: ADEF1

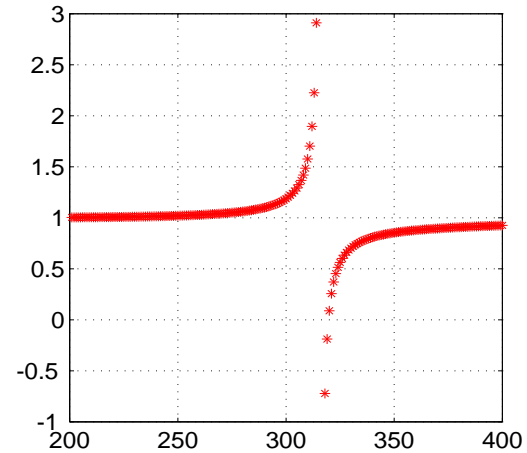
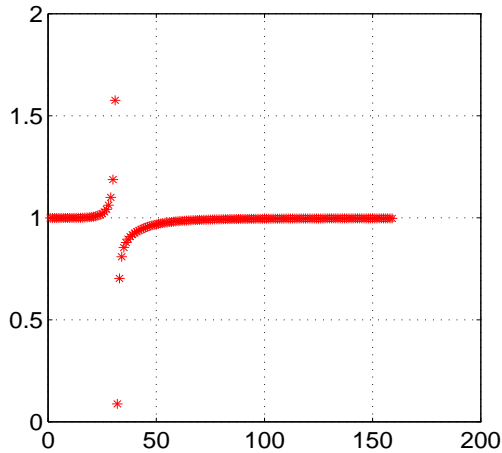
Plotting $\lambda_h(B_{h,2h})$



Spectrum of $B_{h,2h}$ for $k = 100$ and $k = 1000$, 20gp/wl

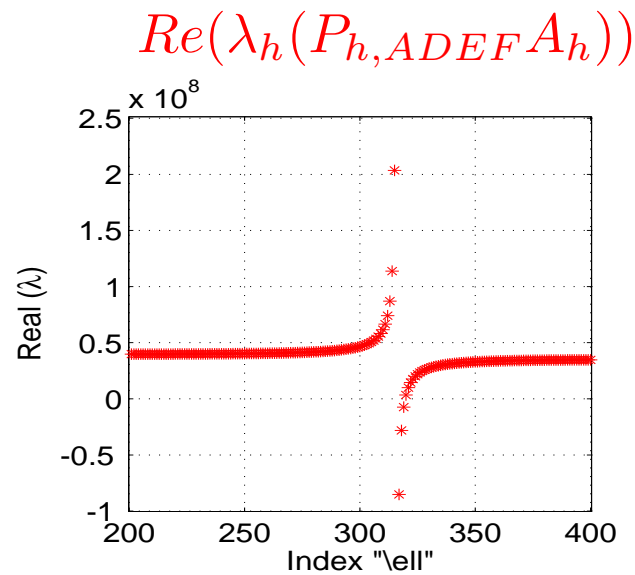
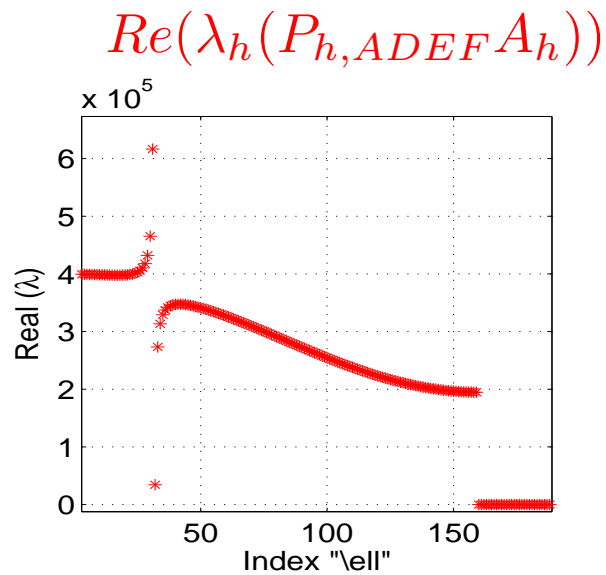
Spectrum insights: ADEF1

Plotting $Re(\lambda_h(P_{h,ADEF1}A_h))$



Real part of the eigenvalues v/s index. $k = 100$ and $k = 1000$, 20gp/wl

Spectrum insights: ADEF1



Real part of the eigenvalues v/s index. $k = 160$, $h = 320$

Deflation: TLKM

Two-Level Krylov Method ^a, if $\hat{A}_h = M_h^{-1} A_h$ and \hat{P}_h is based upon \hat{A}_h (instead A_h)

$$\hat{P}_h = I_h - \hat{A}_h \hat{Q}_h,$$

where

$$\hat{Q}_h = I_{2h}^h \hat{A}_{2h}^{-1} I_h^{2h} \quad \text{and} \quad \hat{A}_{2h} = I_h^{2h} \hat{A}_h I_{2h}^h = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$$

Construction of coarse matrix A_{2h} at level $2h$ costs inversion of preconditioner at level h .

Approximate A_{2h} ?

Ideal

$$A_{2h} = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$$

Practical

$$A_{2h} = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$$

$$A_{2h} \approx \Theta_h M_{2h}^{-1} A_{2h}, \quad \Theta_h = I_h^{2h} I_{2h}^h$$

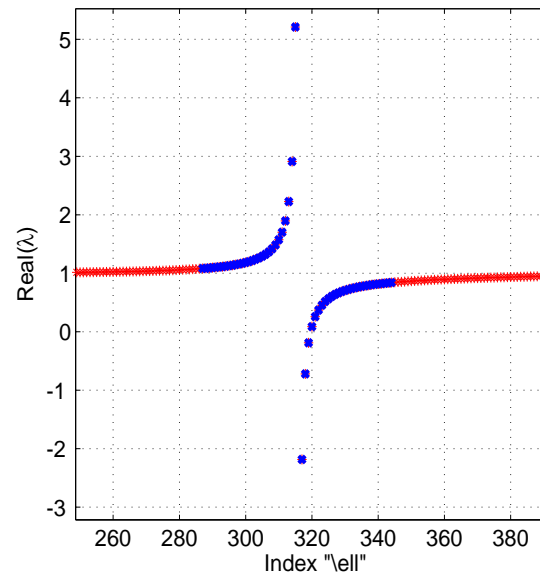
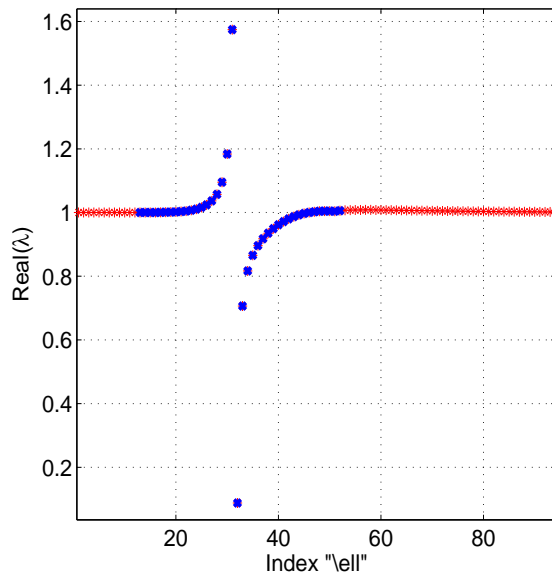
^aErlangga, Y.A and Nabben R., ETNA 2008

Spectral insights: TLKM

Real part of the spectrum of \hat{B}_h where $\hat{B}_h = \hat{P}_h \hat{A}_h$

$$k = 100$$

$$k = 1000$$

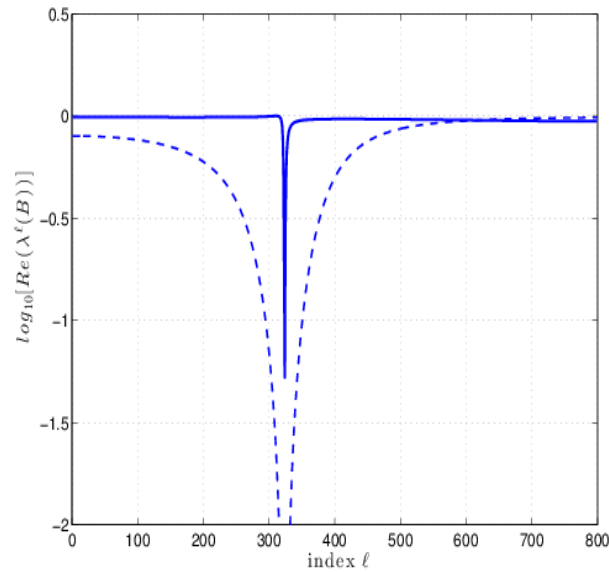
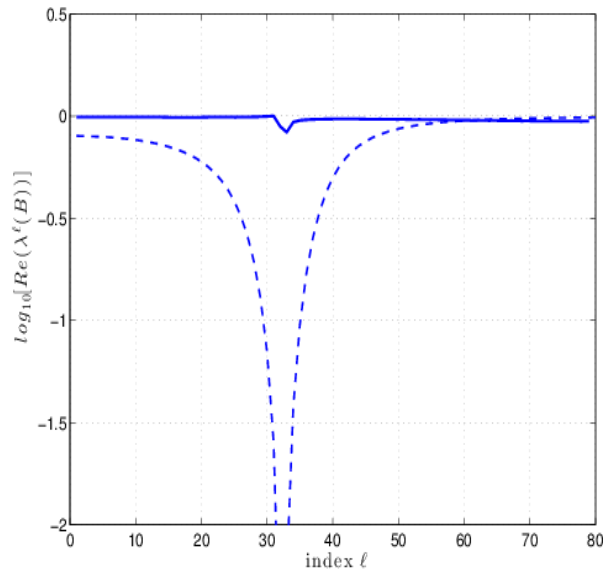


Spectral insights: TLKM

Real part of the eigenvalues of \hat{B}_h vs index. Also the Real part eigenvalues of \hat{A}_h ;

$$k = 100$$

$$k = 1000$$



ADEF1 v TLKM

Differentiating ADEF1 and TLKM, assuming $\lambda_{max} = 1$ and left preconditioning

ADEF1

$$P_{(ADEF1)} = M_h^{-1}(I_h - A_h Q_h) + Q_h$$

Application on $Au = g$

MLKM*

$$P_{(MLKM)} = I_h - \hat{A}_h \hat{Q}_h + \hat{Q}_h$$

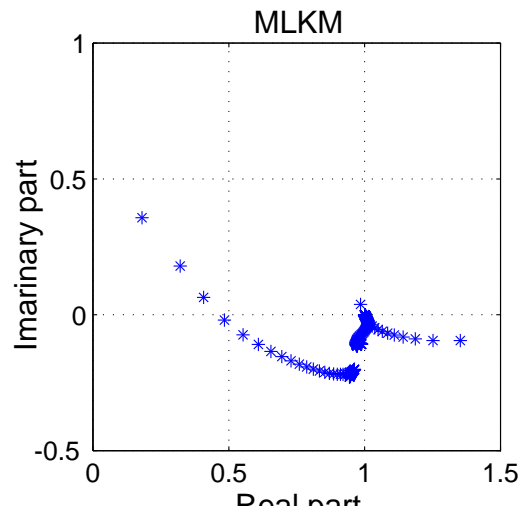
Application on $\hat{A}u = \hat{g}$

Fourier Analysis

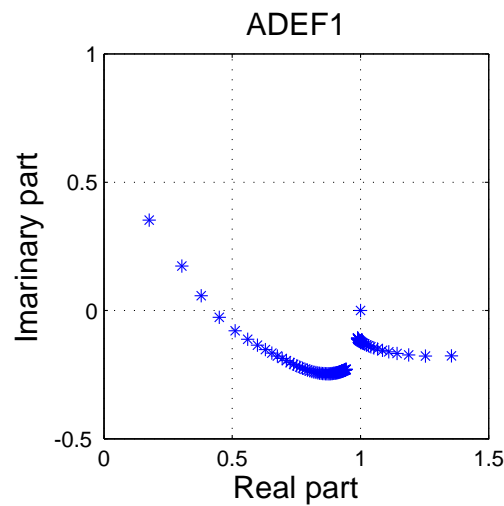
Spectrum of Helmholtz preconditioned by MLKM and ADEF1;

$k = 160$ and 10 gp/wl

TLKM



ADEF1

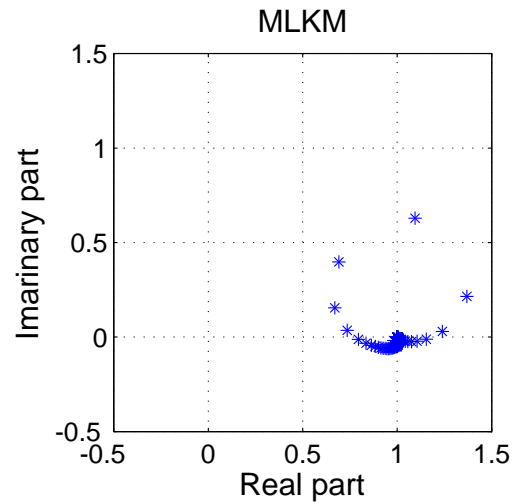


Fourier Analysis

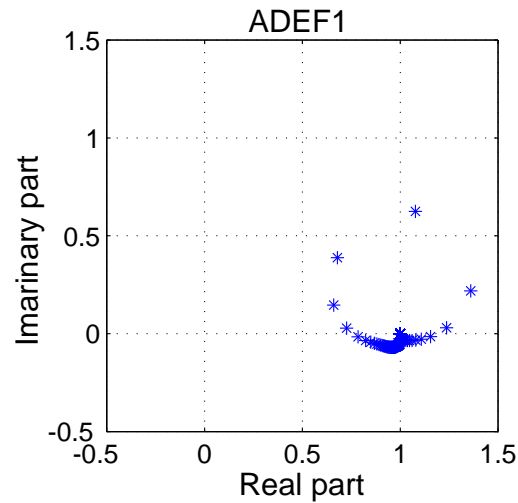
Spectrum of Helmholtz preconditioned by TLKM and ADEF1;

$k = 160$ and 20 gp/wl

TLKM



ADEF1



Cost comparison

Application cost per iteration at two levels

For some vector v ,

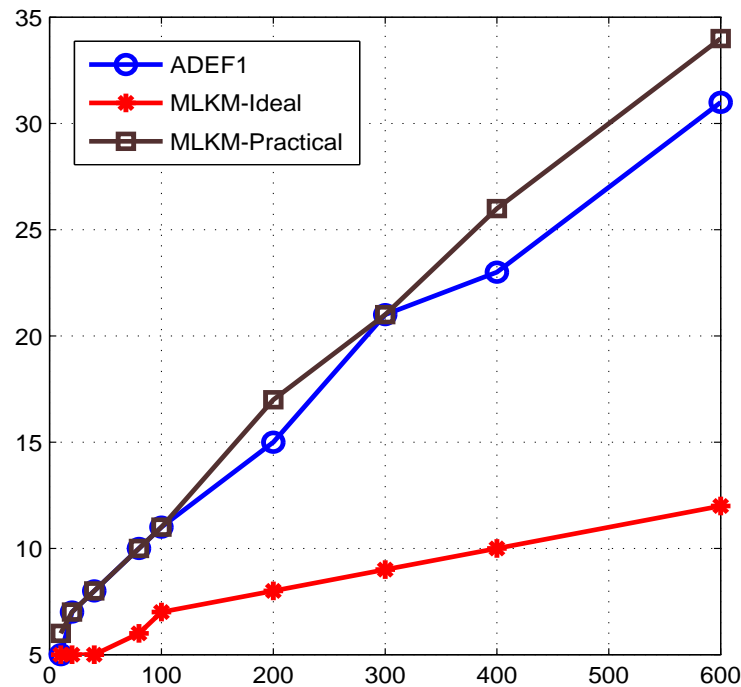
	ADEF1	TLMG
$A_h v$	1	1
$M_h^{-1} v$	1	2
$Q_h v: I_h^{2h} v$	1	1
$Q_h v: I_{2h}^h v$	1	1
$Q_h v: A_{2h}^{-1} v$	1	1
$Q_h v: M_{2h}^{-1} v$	0	1
$\Theta_h v$	0	1

Numerical results

One Dimensional Helmholtz with Som. BCs.

Wave number against Krylov iterations

Two level solver

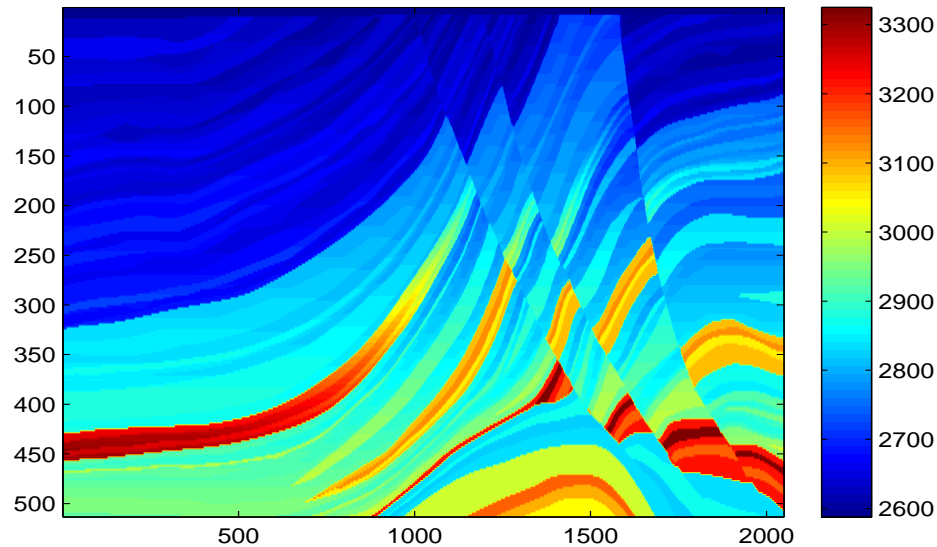


Comparison of number of iterations by [ADEF1](#) and [MLKM](#).

Adapted Marmousi Problem

Reduced velocity contrast: $2587 \leq c(x, y) \leq 3325$

Adapted geometry convenient for geometric vectors.



Results

Marmousi Problem: Solve time and iterations

Frequency f	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
1	1.25	5.06	13	7
10	9.63	9.35	106	13
20	70.45	57.47	181	21
40	522.90	424.74	333	38

Results

Marmousi Problem: Solve time and iterations; discretization 20 gp/wl

Frequency f	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
$f = 1$	1.23	5.08	13	7
$f = 10$	40.01	21.83	106	8
$f = 20$	280.08	131.30	177	12
$f = 40$	20232.6	3997.7	340	21

Numerical results

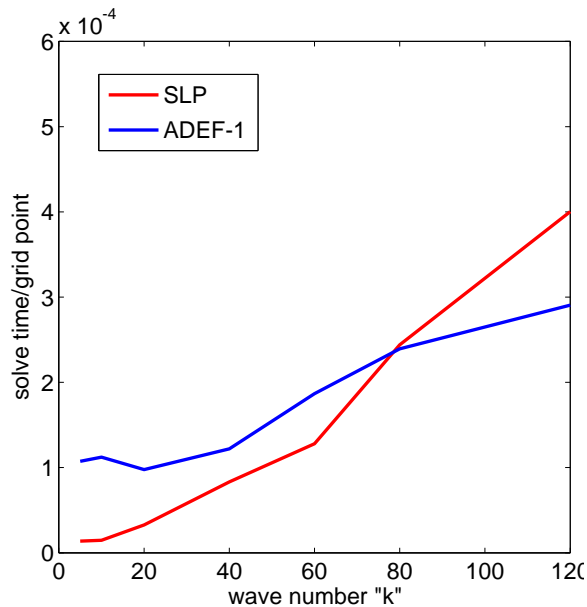
Three Dimensional Helmholtz on unit cube domain with Sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.3125$

Wave number	Solve Time		Iterations	
k	SLP-F	ADEF1-F	SLP-F	ADEF1-F
5	0.04	0.32	7	8
10	0.48	2.32	9	9
20	8.14	17.28	20	9
40	228.29	155.52	70	10
60	1079.99	607.45	97	11

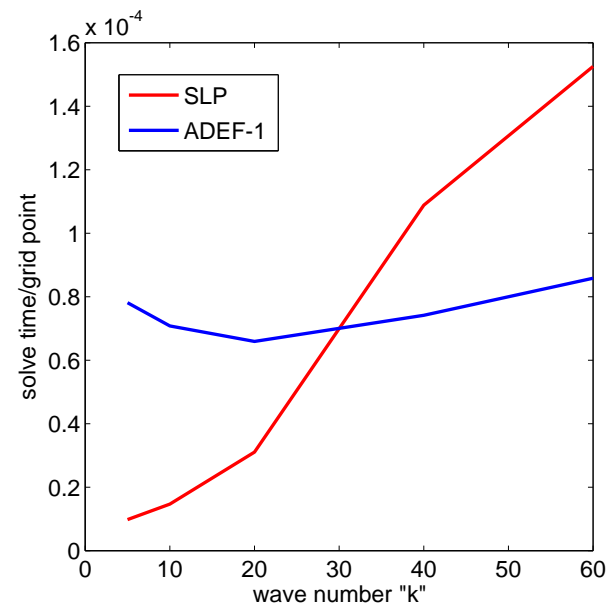
Results

Solve time per grid point.

10gp/wl



20gp/wl



Numerical results

Three Dimensional Layered Helmholtz on unit cube domain with Sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.625$

Wave number k	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
5	0.09	0.24	9	11
10	1.07	1.94	15	12
20	16.70	18.89	32	16
30	73.82	78.04	43	21
40	1304.2	214.7	331	24
60	-	989.5	500+	34

Numerical results

Three Dimensional Layered Helmholtz on unit cube domain with Sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.3125$

Wave number k	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
5	0.6	1.4	9	9
10	7.5	10.04	14	9
20	324.1	79.2	72	9
30	3810.9	361.7	285	11

Numerical results

Algebraic deflation vectors ?

- FEM regular mesh triangular element discretization.
- Algebraically constructed deflation; AMG cycle.
- ADEF1 preconditioner.
- Comparison with FDM.
- Algebraic vectors proceed the coarsening slower than geometric.
- Mesh is refined enough till satisfactory the wavelength resolution.

Numerical results

Iteration count comparison of Geometric(FDM) and Algebraic(FEM) deflation

Solver	FEM/ FDM	k=10	20	40	80	120	160	200
CSLP-D	FEM	15	30	57	108	157	204	252
CSLP-F	FEM	22	43	72	128	178	232	278
ADEF1-F	FEM	7	8	10	16	19	24	27
	FDM	10	11	15	24	32	41	51
Two-Lev	FEM	6	8	10	15	20	26	32
	FDM	7	10	14	23	37	61	87
ADEF1-V	FEM	22	40	66	118	166	214	258
	FDM	16	27	58	116	177	235	292
ADEF1-D	FEM	6	8	10	15	19	24	27
	FDM	7	10	14	21	29	36	43

Numerical results

Solve time comparison of Geometric(FDM) and Algebraic(FEM) deflation

k	Solver	SLP-D	SLP-F	ML-DEF		TL-DEF	
				FEM	FDM	FEM	FDM
10		0.02	0.05	0.25	0.06	0.02	0.01
20		0.07	0.16	0.85	1.6	0.05	0.03
40		0.57	0.85	2.4	4.5	0.32	0.27
80		5.8	6.33	15.2	15.7	2.46	2.17
120		22.6	21.8	38.3	28.2	8.4	8.8
160		59.6	55.7	81.5	70.1	21.4	27.9
200		130.5	115.9	144.5	103.9	43.8	67.8

Conclusion and Discussion

- Near null space modes in A_h persist. Same time extraordinary gain in Krylov iterations. Ritz testing in progress.
- How to treat near-null space modes in coarser operators ?
- FEM discretization and algebraic deflation vectors in 3-dimension failed. Mass matrix NOT diagonal, it has negative entries off-diagonal.
- Flexible in choosing larger imaginary shift in CSLP. Reported!
- Adapted coarse grid operator. Work in progress!
- Different shifts in SLP at different levels. Future!

References

- Y.A. Erlangga and R. Nabben. On a multilevel Krylov method for the Helmholtz equation preconditioned by shifted Laplacian. ETNA, 2008.
- M.B. van Gijzen, Y.A. Erlangga and C. Vuik. Spectral analysis of the discrete Helmholtz operator preconditioned with a shifted Laplacian. SIAM J.of Sc. Comp. 2007.
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- A.H. Sheikh, D. Lahaye and C. Vuik. On the convergence of shifted Laplace preconditioner combined with multi-grid deflation. NLAA Volume 20, Issue 4, pages 645-662, August 2013
- A.H. Sheikh Development Of The Helmholtz Solver Based On A Shifted Laplace Preconditioner And A Multigrid Deflation Technique, Thesis, TU Delft, 2015
- A.H. Sheikh, D. Lahaye and C. Vuik Accelerating the Shifted Laplace Preconditioner for the Helmholtz Equation by Multilevel Deflation, submitted JCP, 2015
- http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_helmholtz.html

Thank you!