Efficient and robust preconditioners for high Reynolds number laminar flows

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Motivation

Achievements on preconditioning Reynolds-Averaged Navier-Stokes equations:

- New augmented Lagrangian (AL) preconditioner utilizes the approximation of the Schur complement from the SIMPLE preconditioner.
- A significantly improved convergence rate.

On preconditioning laminar Navier-Stokes equations, this work is motivated by the following questions:

- Does the new AL preconditioner still work efficiently?
- Does the utilization of other available Schur complement approximations in the new AL preconditioner result in a better performance?
- If so, which is the "optimal" one?
- what is the effect of the Reynolds number, mesh anisotropy and refinement on the optimal choice?

Incompressible Navier-Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{on } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{on } \Omega. \end{aligned}$$

- $\mathbf{u}$: the velocity, p: the pressure, ν: the kinematic viscosity.
- $\Omega \in \mathcal{R}^{2 \text{ or } 3}$ is a bounded domain with the boundary $\partial\Omega$

$$\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega_D, \;\; \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \text{ on } \partial\Omega_N$$

Linear system

$$\begin{bmatrix} A & B^T \\ B & -\frac{1}{\nu}C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix} \quad \text{with} \quad \mathcal{A} := \begin{bmatrix} A & B^T \\ B & -\frac{1}{\nu}C \end{bmatrix},$$

- A : the diagonal blocks A_{ii} correspond to the convection diffusion operator.
- $B, \ B^T$: the divergence and gradient matrices.
- C : the stabilization matrix for the Q_1-Q_1 discretization [Dohrman 2012].

  $$C^{(macro)} = M^{(macro)} - qq^T|\Box_k|,$$

  where $M^{(macro)}$ is the 4×4 macroelement mass matrix and $q = [1/4, 1/4, 1/4, 1/4]^T$ is the local averaging operator.
- $\mathcal{A}$: sparse and non-symmetric.

[Dohrman 2012] C.R. Dohrmann and P.B. Bochev. A stabilized finite element method for the Stokes problem based on polynomial pressure projections. *International Journal for Numerical Methods in Fluids*, 46:183201, 2012.

Block structured preconditioners

The block $\mathcal{LDU}$ decomposition of $\mathcal{A}$ is

$$\mathcal{A} = \mathcal{LDU} = \begin{bmatrix} A & B^T \\ B & -\frac{1}{\nu}C \end{bmatrix} = \begin{bmatrix} I_1 & O \\ BA^{-1} & I_2 \end{bmatrix} \begin{bmatrix} A & O \\ O & S \end{bmatrix} \begin{bmatrix} I_1 & A^{-1}B^T \\ O & I_2 \end{bmatrix},$$

$S = -(\nu^{-1}C + BA^{-1}B^T)$ is the so-called Schur complement.

The block structured preconditioners $\mathcal{P}_L$ and $\mathcal{P}_U$

$$\mathcal{P}_L = \mathcal{LD} = \begin{bmatrix} A & O \\ B & \widetilde{S} \end{bmatrix}, \qquad \mathcal{P}_U = \mathcal{DU} = \begin{bmatrix} A & B^T \\ O & \widetilde{S} \end{bmatrix}.$$

- solve the velocity subsystem with A,
- solve the pressure subsystem with $\widetilde{S} \approx S$.

How to find a spectrally equivalent and cheap approximation of S.

Block structured preconditioners

For the stabilized system, the available Schur complement approximations are:

- The pressure convection-diffusion operator $\widetilde{S}_{PCD}$:

$$\widetilde{S}_{PCD} = -L_p A_p^{-1} M_p,$$

  M_p : the pressure mass matrix, A_p : the pressure convection-diffusion operator, L_p : the pressure Laplacian operators.

- The least-square commutator $\widetilde{S}_{LSC}$:

$$\widetilde{S}_{LSC} = -(B\widehat{M}_u^{-1}B^T + C_1)(B\widehat{M}_u^{-1}A\widehat{M}_u^{-1}B^T + C_2)^{-1}(B\widehat{M}_u^{-1}B^T + C_1),$$

  $\widehat{M}_u$: the diagonal approximation of the velocity mass matrix M_u and

$$C_1^{(macro)} = \frac{1}{|\square_k|} \cdot C^{(macro)}, \quad C_2^{(macro)} = \frac{\nu}{|\square_k|^2} \cdot C^{(macro)},$$

- The approximation $\widetilde{S}_{SIMPLE}$ from the SIMPLE preconditioner:

$$\widetilde{S}_{SIMPLE} = -(\nu^{-1}C + B\mathrm{diag}(A)^{-1}B^T).$$

- The augmented Lagrangian preconditioner (more details on the following slides)

The augmented Lagrangian preconditioner

System $\begin{bmatrix} A & B^T \\ B & -\frac{1}{\nu}C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix}$ is transformed into

$$\begin{bmatrix} A_\gamma & B_\gamma^T \\ B & -\frac{1}{\nu}C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f}_\gamma \\ g \end{bmatrix} \quad \text{with} \quad \mathcal{A}_\gamma := \begin{bmatrix} A_\gamma & B_\gamma^T \\ B & -\frac{1}{\nu}C \end{bmatrix},$$

- $A_\gamma = A + \gamma B^T W^{-1} B$, $B_\gamma^T = B^T - \gamma/\nu B^T W^{-1} C$ and $\mathbf{f}_\gamma = \mathbf{f} + \gamma B^T W^{-1} g$
- $\gamma > 0$ and W are scalar and nonsingular matrix parameters.
- the Schur complement of $\mathcal{A}_\gamma$ is $S_\gamma = -(\nu^{-1} C + B A_\gamma^{-1} B_\gamma^T)$.

The augmented Lagrangian preconditioner

The ideal AL preconditioner $\mathcal{P}_{IAL}$ and its variant, i.e. the modified AL preconditioner $\mathcal{P}_{MAL}$ are based on the block $\mathcal{DU}$ decomposition of $\mathcal{A}_\gamma$

$$\mathcal{P}_{IAL} = \begin{bmatrix} A_\gamma & B_\gamma^T \\ O & \widetilde{S}_\gamma \end{bmatrix} \text{ and } \mathcal{P}_{MAL} = \begin{bmatrix} \widetilde{A}_\gamma & B_\gamma^T \\ O & \widetilde{S}_\gamma \end{bmatrix},$$

where $\widetilde{S}_\gamma$ and $\widetilde{A}_\gamma$ denote the approximations of S_γ and A_γ, respectively.

$$A_\gamma = \begin{bmatrix} A_1 + \gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & A_1 + \gamma B_2^T W^{-1} B_2 \end{bmatrix} \text{ (\textit{coupling of} } B_i^T B_j \ (i \neq j))$$

$$\widetilde{A}_\gamma = \begin{bmatrix} A_1 + \gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ O & A_1 + \gamma B_2^T W^{-1} B_2 \end{bmatrix} \text{ (\textit{no coupling of} } B_i^T B_j \ (i \neq j))$$

The new Schur approximation in the AL preconditioner

The novel approximation is based on:

Lemma

Assuming that all the relevant matrices are invertible, then the inverse of S_γ *is given by*

$$S_\gamma^{-1} = S^{-1}(I - \gamma \widehat{C} W^{-1}) - \gamma W^{-1},$$

where $S = -(\widehat{C} + BA^{-1}B^T)$ *denotes the Schur complement of the original system with* $\mathcal{A}$ *and* $\widehat{C}$ *is defined as* $\widehat{C} = \nu^{-1} C$*.*

Proof.

For the proof we refer to [He 2017]. □

[He 2017] X. He, C. Vuik and C. Klaij. Block preconditioners for the incompressible Navier-Stokes equations discretized by a finite volume method. *Journal of Numerical Mathematics*, 25:89-105, 2017.

New Schur approximation in the AL preconditioner

New option: $\widetilde{S}_{\gamma\ \text{new}}^{-1} = \widetilde{S}^{-1}(I - \gamma \widehat{C} W^{-1}) - \gamma W^{-1}$.

Substituting the Schur approximations $\widetilde{S}$ for the original system with $\mathcal{A}$, three variants of $\widetilde{S}_{\gamma\ \text{new}}$ are derived as

- $\widetilde{S}_{\gamma\ PCD}^{-1} = \widetilde{S}_{PCD}^{-1}(I - \gamma \widehat{C} W^{-1}) - \gamma W^{-1}$,
- $\widetilde{S}_{\gamma\ LSC}^{-1} = \widetilde{S}_{LSC}^{-1}(I - \gamma \widehat{C} W^{-1}) - \gamma W^{-1}$,
- $\widetilde{S}_{\gamma\ SIMPLE}^{-1} = \widetilde{S}_{SIMPLE}^{-1}(I - \gamma \widehat{C} W^{-1}) - \gamma W^{-1}$.

Comments:

- W : the diagonal approximation of the pressure mass matrix, i.e. $W = \widehat{M}_p = \text{diag}(M_p)$, it is trivial to obtain its inverse.
- The complexity of applying $\widetilde{S}_{\gamma\ \text{new}}$ is mainly focused on solving the system with $\widetilde{S}$.
- No contradictory requirement of the value of γ compared to the old Schur approximation.

Ole Schur approximation in the AL preconditioner

Old option 1: $W_1 = \gamma \widehat{C} + M_p$ and $\widetilde{S}_{\gamma \text{ old}} = -(\widehat{C} + \gamma^{-1} M_p)$.

Choosing $W_1 = \gamma \widehat{C} + M_p$ and substituting W_1 into $S_\gamma^{-1} = S^{-1}(I - \gamma \widehat{C} W^{-1}) - \gamma W^{-1}$, leads to

$$S_\gamma^{-1} = (\gamma^{-1} S^{-1} M_p - I)(\widehat{C} + \gamma^{-1} M_p)^{-1}$$

For large values of γ such that $\| \gamma^{-1} S^{-1} M_p \| \ll 1$ we can approximate S_γ by

$$\widetilde{S}_{\gamma \text{ old}} = -(\widehat{C} + \gamma^{-1} M_p).$$

Comment:

- $W_1 = \gamma \widehat{C} + M_p$ is not a practical option since its inverse is needed in the AL transformation.

Old Schur approximation in the AL preconditioner

Old option 2: $W = \widehat{M}_p = \text{diag}(M_p)$ and $\widetilde{S}_{\gamma \text{ old}} = -(\widehat{C} + \gamma^{-1} M_p)$ [Benzi 2011]

Comments:

- The approximation $\widetilde{S}_{\gamma \text{ old}}$ is obtained if and only if $W_1 = \gamma \widehat{C} + M_p$ and large values of γ are chosen.
- However, $W = \widehat{M}_p$ is spectrally equivalent to $W_1 = \gamma \widehat{C} + M_p$ only when γ is small.
- it is contradictory to tune the value of γ so that W and $\widetilde{S}_{\gamma \text{ old}}$ could be simultaneously obtained.

[Benzi 2011] M. Benzi, M.A. Olshanskii, Z. Wang. Modified augmented Lagrangian preconditioners for the incompressible Navier-Stokes equations. *Int. J. Numer. Meth. Fluids.*, 66:486-508, 2011.

Comparison between the Schur approximations in AL

With $W = \widehat{M}_p = \mathrm{diag}(M_p)$ and $\widehat{C} = \nu^{-1} C$, the Schur approximations in $\mathcal{P}_{MAL}$ are

1. $\widetilde{S}_{\gamma\ PCD}^{-1} = \widetilde{S}_{PCD}^{-1}(I - \gamma \widehat{C} \widehat{M}_p^{-1}) - \gamma \widehat{M}_p^{-1}$,
2. $\widetilde{S}_{\gamma\ LSC}^{-1} = \widetilde{S}_{LSC}^{-1}(I - \gamma \widehat{C} \widehat{M}_p^{-1}) - \gamma \widehat{M}_p^{-1}$,
3. $\widetilde{S}_{\gamma\ SIMPLE}^{-1} = \widetilde{S}_{SIMPLE}^{-1}(I - \gamma \widehat{C} \widehat{M}_p^{-1}) - \gamma \widehat{M}_p^{-1}$,
4. $\widetilde{S}_{\gamma\ \mathrm{old}} = -(\widehat{C} + \gamma^{-1} M_p)$.

where

1. $\widetilde{S}_{PCD} = -L_p A_p^{-1} M_p$,
2. $\widetilde{S}_{LSC} = -(B\widehat{M}_u^{-1}B^T + C_1)(B\widehat{M}_u^{-1}A\widehat{M}_u^{-1}B^T + C_2)^{-1}(B\widehat{M}_u^{-1}B^T + C_1)$,
3. $\widetilde{S}_{SIMPLE} = -(\widehat{C} + B\mathrm{diag}(A)^{-1}B^T)$.

Comparison between the Schur approximations in AL

Table: Pressure sub-system 'mass-p' with $\widetilde{S}_\gamma$ in $\mathcal{P}_{MAL}$ and the systems involved therein.

'mass-p' with $\widetilde{S}_{\gamma\ \mathrm{new}}$	'mass-p' with $\widetilde{S}$	systems involved in $\widetilde{S}$
$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{PCD}$	L_p and M_p
$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{LSC}$	$(B\widehat{M}_u^{-1}B^T + C_1)$ twice
$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{SIMPLE}$	$\widehat{C} + B\mathrm{diag}(A)^{-1}B^T$
'mass-p' with $\widetilde{S}_{\gamma\ \mathrm{old}}$	–	systems involved in $\widetilde{S}_{\gamma\ \mathrm{old}}$
$\widetilde{S}_{\gamma\ \mathrm{old}}$	–	$\widehat{C} + \gamma^{-1}M_p$

- At each Krylov iteration the costs of applying $\mathcal{P}_{MAL}$ with $\widetilde{S}_{\gamma\ LSC}$ and $\widetilde{S}_{\gamma\ PCD}$ are roughly the same and two times of that using $\widetilde{S}_{\gamma\ SIMPLE}$ and $\widetilde{S}_{\gamma\ \mathrm{old}}$.

Numerical experiments

(1) **Flow over a finite flat plate (FP)**.
Domain: $\Omega = (-1, 5) \times (-1, 1)$
Reynolds number: $Re = U_{ref} L_{ref}/\nu = \{10^2, 10^3, 10^4, 10^5\}$ ($U_{ref} = 1,\ L_{ref} = 5$).
The uniform Cartesian grid: $12 \times 2^n \cdot 2^n$ cells
The stretched grid: applying the following stretching function in the y-direction:

$$y = \frac{(b+1) - (b-1)c}{(c+1)}, \quad c = (\frac{b+1}{b-1})^{1-\bar{y}}, \quad \bar{y} = 0, 1/n, 2/n, ...1, b = 1.01.$$

(2) **Flow over backward facing step (BFS)**.
Domain: $\Omega = (-1, 5) \times (-1, 1)$ without the square $(-1, 0) \times (-1, 0)$
Reynolds number: $Re = U_{ref} L_{ref}/\nu = \{10^2, 10^3\}$ ($U_{ref} = 1,\ L_{ref} = 2$).
The uniform Cartesian grid: $11 \times 2^n \cdot 2^n$ cells

(3) **Lid driven cavity (LDC)**.
Domain: the square cavity $(-1, 1)^2$.
Reynolds number: $Re = U_{ref} L_{ref}/\nu = \{10^2, 10^3, 10^4\}$ ($U_{ref} = 1,\ L_{ref} = 2$).
The uniform Cartesian grid: $2^n \cdot 2^n$ cells
The stretched grid: applying the following stretching function in both directions:

$$x = \frac{(b+2a)c - b + 2a}{(2a+1)(1+c)}, c = (\frac{b+1}{b-1})^{\frac{\bar{x}-a}{1-a}}, \bar{x} = 0, 1/n, 2/n, ..., 1, a = 0.5, b = 1.01.$$

Numerical experiments

Numerical evaluations are classified into four categories as follows.

(C1) **On small Reynolds number and uniform grid**
FP, BFS and LDC cases: $Re = 10^2$ and uniform Cartesian grid.

(C2) **On moderate Reynolds number and uniform grid**
FP, BFS and LDC cases: $Re = 10^3$ and uniform Cartesian grid.

(C3) **On moderate Reynolds numbers and stretched grid**
FP and LDC cases: $Re = 10^3$ and stretched grids.

(C4) **On large Reynolds numbers and stretched grid**
LDC case:$Re = 10^4$ and stretched grid.
FP case: $Re = \{10^4, 10^5\}$ and stretched grid.

Numerical experiments

- The linear system is obtained at the middle step of the whole nonlinear iterations.
- The relative stopping tolerance to solve the linear system by GMRES is 10^{-8}.
- The momentum and pressure sub-systems are directly solved.

On small Reynolds number and uniform grid

Table: **Re** $= \mathbf{10^2}$ and **uniform grid**: the number of GMRES iterations to apply $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$. The corresponding γ_{opt} is in parentheses.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \text{old}}$
FP case:				
$n = 5$	26(1.e-1)	17(8.e-2)	43(2.e-1)	38(2.e-1)
$n = 6$	25(1.e-1)	25(8.e-2)	67(2.e-1)	38(2.e-1)
$n = 7$	25(1.e-1)	26(8.e-2)	100(2.e-1)	38(2.e-1)
BFS case:				
$n = 5$	34(2.e-2)	17(2.e-2)	42(1.e-1)	36(1.e-1)
$n = 6$	42(3.e-2)	21(2.e-2)	60(1.e-1)	36(1.e-1)
$n = 7$	45(3.e-2)	22(2.e-2)	87(1.e-1)	36(1.e-1)
LDC case:				
$n = 6$	17(2.e-2)	17(2.e-2)	34(1.e-1)	19(1.e-1)
$n = 7$	18(2.e-2)	20(2.e-2)	48(1.e-1)	19(1.e-1)
$n = 8$	18(2.e-2)	22(2.e-2)	63(1.e-1)	19(1.e-1)

- Except $\widetilde{S}_{\gamma\ SIMPLE}$, the other Schur approximations results in mesh independence.
- $\widetilde{S}_{\gamma\ LSC}$ results in the minimal number of iterations.

On small Reynolds number and uniform grid

Assumption: at each Krylov iteration the costs of applying $\mathcal{P}_{MAL}$ with $\widetilde{S}_{\gamma\ LSC}$ and $\widetilde{S}_{\gamma\ PCD}$ are the same and two times of that using $\widetilde{S}_{\gamma\ SIMPLE}$ and $\widetilde{S}_{\gamma\ \mathrm{old}}$.

Table: **Re** $= \mathbf{10^2}$ and **uniform grid**: the total costs of applying $\mathcal{P}_{MAL}$ with different $\widetilde{S}_{\gamma}$ and the corresponding γ_{opt} on the finest uniform Cartesian grid.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \mathrm{old}}$
FP case:	50	52	100	38
BFS case:	90	44	87	36
LDC case:	36	44	63	19

- $\widetilde{S}_{\gamma\ \mathrm{old}}$ results in the minimal computational cost, which can be expected on finer grids due to the mesh independence.

On moderate Reynolds number and uniform grid

Table: **Re** $= \mathbf{10^3}$ and **uniform grid**: the number of GMRES iterations to apply $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$. The corresponding γ_{opt} is in parentheses.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \text{old}}$
FP case:				
$n = 5$	54(8.e-3)	29(8.e-3)	34(2.e-2)	76(6.e-2)
$n = 6$	55(8.e-3)	18(8.e-3)	51(2.e-2)	90(6.e-2)
$n = 7$	56(8.e-3)	17(8.e-3)	99(2.e-2)	95(6.e-2)
BFS case:				
$n = 5$	66(4.e-3)	45(3.e-3)	49(1.e-2)	71(3.e-2)
$n = 6$	63(4.e-3)	27(3.e-3)	77(1.e-2)	76(3.e-2)
$n = 7$	65(3.e-3)	29(3.e-3)	142(1.e-2)	84(3.e-2)
LDC case:				
$n = 6$	30(4.e-3)	54(1.e-3)	66(7.e-3)	36(2.e-2)
$n = 7$	28(4.e-3)	29(4.e-3)	52(1.e-2)	42(2.e-2)
$n = 8$	29(4.e-3)	29(4.e-3)	85(1.e-2)	48(2.e-2)

- $\widetilde{S}_{\gamma\ PCD}$ and $\widetilde{S}_{\gamma\ LSC}$ result in the mesh independence.
- $\widetilde{S}_{\gamma\ LSC}$ results in the minimal number of iterations.

On moderate Reynolds number and uniform grid

Table: **Re** $= \mathbf{10^3}$ and **uniform grid**: the total costs of applying $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$ and the corresponding γ_{opt} on the finest uniform Cartesian grid.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \text{old}}$
FP case:	112	34	99	95
BFS case:	130	58	142	84
LDC case:	58	58	85	48

- $\widetilde{S}_{\gamma\ LSC}$ results in the minimal computational cost, which can be expected on finer grids due to the mesh independence.

On moderate Reynolds number and stretched grid

Table: **Re** $= \mathbf{10^3}$ and **stretched grid**: the number of GMRES iterations to apply $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$. The corresponding γ_{opt} is in parentheses.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \text{old}}$
FP case:				
$n = 5$	59(8.e-3)	90(7.e-3)	37(2.e-2)	69(6.e-2)
$n = 6$	66(8.e-3)	89(7.e-3)	63(2.e-2)	85(6.e-2)
$n = 7$	62(8.e-3)	117(6.e-3)	119(2.e-2)	92(6.e-2)
LDC case:				
$n = 6$	65(2.e-3)	98(2.e-3)	57(7.e-3)	69(1.e-2)
$n = 7$	41(2.e-3)	58(2.e-3)	46(7.e-3)	40(1.e-2)
$n = 8$	38(2.e-3)	84(2.e-3)	75(7.e-3)	54(1.e-2)

Table: **Re** $= \mathbf{10^3}$ and **stretched grid**: the total costs of applying $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$ and the corresponding γ_{opt} on the finest stretched grid.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \text{old}}$
FP case:	124	234	119	92
LDC case:	76	168	75	54

- $\widetilde{S}_{\gamma\ PCD}$ results in the mesh independence and minimal number of iterations, which will lead to less computational costs in total on finer grids.

On large Reynolds number and stretched grid

Table: **Re** $= \mathbf{10^4}$ and **stretched grid**: the number of GMRES iterations to apply $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$. The corresponding γ_{opt} is in parentheses.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \mathrm{old}}$
FP case:				
$n = 5$	363(8.e-4)	369(6.e-4)	35(2.e-3)	93(1.e-2)
$n = 6$	334(8.e-4)	336(6.e-4)	53(3.e-3)	128(2.e-2)
$n = 7$	346(8.e-4)	374(6.e-4)	83(4.e-3)	192(2.e-2)
LDC case:				
$n = 6$	113(3.e-4)	97(2.e-4)	34(1.e-3)	46(5.e-3)
$n = 7$	143(3.e-4)	235(2.e-4)	45(1.e-3)	65(5.e-3)
$n = 8$	159(4.e-4)	309(2.e-4)	80(2.e-3)	106(5.e-3)

Table: **Re** $= \mathbf{10^4}$ and **stretched grid**: the total costs of applying $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$ and the corresponding γ_{opt} on the finest stretched grid.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \mathrm{old}}$
FP case:	692	748	83	192
LDC case:	318	618	80	106

- No Schur approximation results in mesh indepdence. However, $\widetilde{S}_{\gamma\ SIMPLE}$ results in the minimal number of iterations and computational costs.

On large Reynolds number and stretched grid

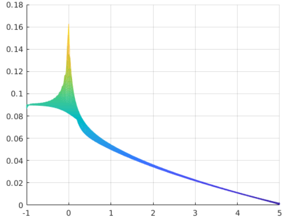
Table: **FP** and $\mathbf{Re = 10^5}$: the number of GMRES iterations and total costs to apply $\mathcal{P}_{MAL}$ with different $\widetilde{S}_\gamma$. The corresponding γ_{opt} is in parentheses.

	$\widetilde{S}_{\gamma\ PCD}$	$\widetilde{S}_{\gamma\ LSC}$	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{\gamma\ \text{old}}$
iterations:				
$n = 5$	1000+	1000+	26(1.e-4)	136(1.e-3)
$n = 6$	1000+	1000+	35(2.e-4)	192(2.e-3)
$n = 7$	1000+	1000+	58(3.e-4)	310(2.e-3)
total costs:				
$n = 7$	2000+	2000+	58	310

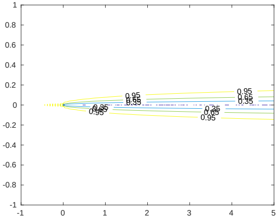
- On the highest Reynolds number of $Re = 10^5$, $\widetilde{S}_{\gamma\ SIMPLE}$ reduces the total computational costs at least five times.

On large Reynolds number and stretched grid

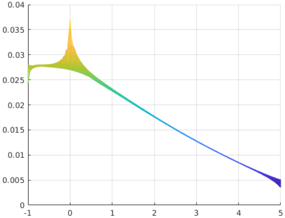
Figure: **FP:** plot of the pressure unknown (left) and equally spaced contours of the horizontal velocity between 0 and 0.95 (right) at different Reynolds numbers.



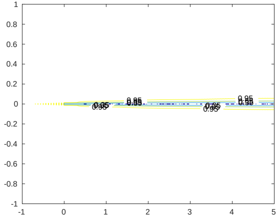
(a) plot of the pressure at $Re = 10^4$



(b) contours of the velocity u at $Re = 10^4$



(c) plot of the pressure at $Re = 10^5$



(d) contours of the velocity u at $Re = 10^5$

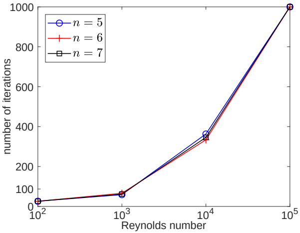
Summary of the Schur approximations in AL

Table: The optimal Schur approximation $\widetilde{S}_{\gamma \text{ opt}}$ in the modified AL preconditioner on varying classes of evaluations.

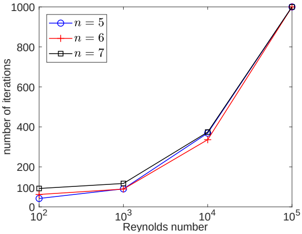
class of evaluations	$\widetilde{S}_{\gamma \text{ opt}}$	mesh independence	problem independence
$Re = 10^2$ and uniform grid	$\widetilde{S}_{\gamma \text{ old}}$	Yes	Yes
$Re = 10^3$ and uniform grid	$\widetilde{S}_{\gamma\ LSC}$	Yes	Yes
$Re = 10^3$ and stretched grid	$\widetilde{S}_{\gamma\ PCD}$	Yes	Yes
$Re \geq 10^4$ and stretched grid	$\widetilde{S}_{\gamma\ SIMPLE}$	No	Yes

Summary of the Schur approximations in AL

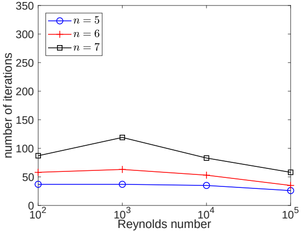
Figure: **FP** and **stretched grid**: plot of the number of GMRES iterations preconditioned by $\mathcal{P}_{MAL}$ at varying Reynolds numbers.



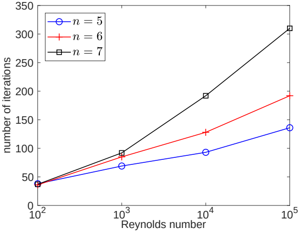
(a) $\mathcal{P}_{MAL}$ with $\widetilde{S}_{\gamma\ PCD}$



(b) $\mathcal{P}_{MAL}$ with $\widetilde{S}_{\gamma\ LSC}$



(c) $\mathcal{P}_{MAL}$ with $\widetilde{S}_{\gamma\ SIMPLE}$



(d) $\mathcal{P}_{MAL}$ with $\widetilde{S}_{\gamma\ \mathrm{old}}$

- $\widetilde{S}_{\gamma\ SIMPLE}$ is robust with respect to the Reynolds number.

Comparison between $\mathcal{P}_{MAL}$ and $\mathcal{P}_U$

The original system and the block structured preconditioner:

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -\frac{1}{\nu} C \end{bmatrix} \text{ with } \mathcal{P}_U = \begin{bmatrix} A & B^T \\ O & \widetilde{S} \end{bmatrix}.$$

The transformed system and the modified AL preconditioner:

$$\mathcal{A}_\gamma = \begin{bmatrix} A_\gamma & B_\gamma^T \\ B & -\frac{1}{\nu} C \end{bmatrix} \text{ with } \mathcal{P}_{MAL} = \begin{bmatrix} \widetilde{A}_\gamma & B_\gamma^T \\ O & \widetilde{S}_\gamma \end{bmatrix}.$$

The relation between the two Schur approximations $\widetilde{S}$ and $\widetilde{S}_\gamma$ are:

$$\widetilde{S}_{\gamma\ \text{new}}^{-1} = \widetilde{S}^{-1}(I - \gamma \widehat{C} W^{-1}) - \gamma W^{-1}, \ W = \widehat{M}_p = \text{diag}(M_p)$$

linear system	preconditioner	Schur complement approximations
transformed system with $\mathcal{A}_\gamma$	$\mathcal{P}_{MAL}$	$\widetilde{S}_{\gamma\ PCD}$, $\widetilde{S}_{\gamma\ LSC}$, $\widetilde{S}_{\gamma\ SIMPLE}$, $\widetilde{S}_{\gamma\ \text{old}}$
original system with $\mathcal{A}$	$\mathcal{P}_U$	$\widetilde{S}_{PCD}$, $\widetilde{S}_{LSC}$, $\widetilde{S}_{SIMPLE}$

Comparison between $\mathcal{P}_{MAL}$ and $\mathcal{P}_U$

Table: $\mathbf{Re} = \mathbf{10^4}$ and **stretched grid**: the number of GMRES iterations to solve the transformed system with $\mathcal{A}_\gamma$ preconditioned by $\mathcal{P}_{MAL}$ and the number of GMRES iterations to solve the original system with $\mathcal{A}$ preconditioned by $\mathcal{P}_U$.

	$\mathcal{P}_{MAL}$ for $\mathcal{A}_\gamma$	$\mathcal{P}_U$ for $\mathcal{A}$		
	$\widetilde{S}_{\gamma\ SIMPLE}$	$\widetilde{S}_{PCD}$	$\widetilde{S}_{LSC}$	$\widetilde{S}_{SIMPLE}$
LDC case:				
$n = 6$	34(1.e-3)	130	147	83
$n = 7$	45(1.e-3)	246	307	119
$n = 8$	80(2.e-3)	364	560	182
FP case:				
$n = 5$	35(2.e-3)	879	661	62
$n = 6$	53(3.e-3)	1000+	599	122
$n = 7$	83(4.e-3)	1000+	809	229

- On large Reynolds number $\widetilde{S}_{\gamma\ \mathrm{opt}} = \widetilde{S}_{\gamma\ SIMPLE}$.
- On large Reynolds number using $\mathcal{P}_{MAL}$ with $\widetilde{S}_{\gamma\ SIMPLE}$ leads to a faster convergence.

Conclusion and future work

Conclusions:

- We propose three variants based on the new method to approximate the Schur complement for the AL preconditioner.
- We determine the optimal Schur complement for every class of tests, which is dependent of the Reynolds number and mesh anisotropy, but problem independent.
- On large Reynolds numbers the utilization of the Schur approximation from the SIMPLE preconditioner in the new approximation of the Schur complement of the AL preconditioner can significantly reduce the number of iterations.

Future work:

- Evaluate the total wall-clock time of the AL preconditioner with the considered Schur approximations.