Preconditioners for the incompressible Navier Stokes equations

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Incompressible Flow Solvers in MATLAB/COMSOL

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Outline

1. Introduction
2. Solution techniques
3. Advanced ILU preconditioner
4. Numerical Experiments
5. Conclusions

1. Introduction

The incompressible Navier Stokes equation

$$-\nu\nabla^2\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega$$

$$\nabla.\mathbf{u} = 0 \quad \text{in} \quad \Omega.$$

u: fluid velocity; p: pressure
$\nu > 0$ is the kinematic viscosity coefficient ($1/Re$).
$\Omega \subset \mathbf{R}^2$ is a bounded domain with boundary conditions:

$$\mathbf{u} = \mathbf{w} \;\; on \;\; \partial\Omega_D, \quad \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \;\; on \;\; \partial\Omega_N.$$

Finite element discretization

Discrete weak formulation

$$X_h \subset (H_0^1(\Omega))^d, \quad M_h \subset L^2(\Omega)$$

Find $\mathbf{u_h} \in X_h$ and $p_h \in M_h$

$$\nu \int_\Omega \nabla \mathbf{u_h} : \nabla \mathbf{v_h} d\Omega + \int_\Omega (\mathbf{u_h}.\nabla \mathbf{u_h}).\mathbf{v_h} d\Omega - \int_\Omega p_h (\nabla.\mathbf{v_h}) d\Omega = \int_\Omega \mathbf{f}.\mathbf{v_h} d\Omega, \quad \forall \mathbf{v_h} \in X_h,$$

$$\int_\Omega q_h (\nabla.\mathbf{u_h}) d\Omega = 0 \quad \forall q_h \in M_h.$$

Matrix notation

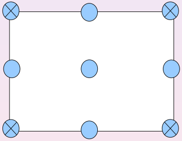
$$A\mathbf{u} + N(\mathbf{u}) + B^T p = \mathbf{f}$$

$$B\mathbf{u} = 0.$$

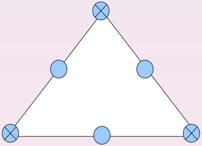
Choice of elements

Brezzi-Babuska condition

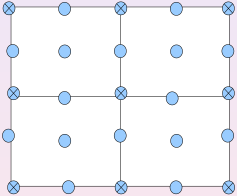
$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{(\nabla . v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h}} \geq \gamma \geq 0.$$



● 9 velocity points (bi-quadratic)
X 4 pressure points(bi-linear)



● 6 velocity points(quadratic)
X 3 pressure points(linear)



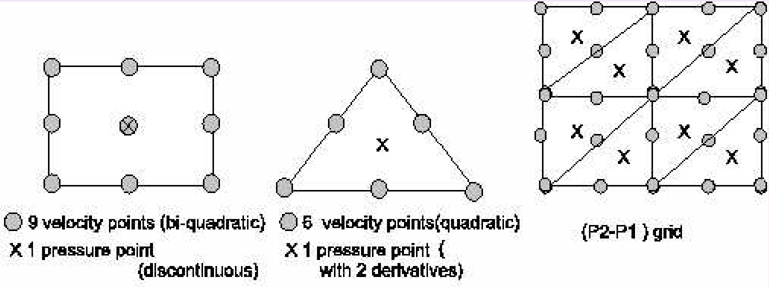
2x2 element mesh

Taylor Hood elements $(Q2 - Q1)$, $(P2 - P1)$ and $(Q2 - Q1)$

Choice of elements

Brezzi-Babuska condition

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{(\nabla . v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h}} \geq \gamma \geq 0.$$

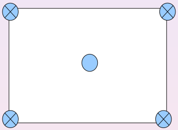


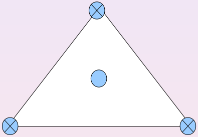
Crouzeix Raviart $(Q2 - P0)$, $(P2^+ - P1)$ and $(P2^+ - P1)$

Choice of elements

Brezzi-Babuska condition

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{(\nabla . v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h}} \geq \gamma \geq 0.$$





● 5 velocity points (bi-linear + bubble at centroid)
X 4 pressure points (bi-linear)

● 4 velocity points (linear + bubble at centroid)
X 3 pressure points (linear)

Taylor Hood mini elements $Q_1^+ - Q_1$ and $P_1^+ - P_1$

Software packages

IFISS

- Incompressible Flow Iterative Solution Software
- Silvester, Elman, Ramage, Wathen
- Matlab, nice for experiments
- academic, 2D problems only
- modern block triangular preconditioners

Sepran

- Sepran = Segal + Praagman
- FORTRAN package, industrial and academic problems
- 1, 2, 3 Dimensional problems
- Complex geometries
- Taylor Hood and Raviart Thomas elements are implemented

Linearization

Stokes problem

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}$$

$$\nabla.\mathbf{u} = 0$$

Picard's method

$$-\nu\Delta\mathbf{u}^{(k+1)} + (\mathbf{u}^{(k)}.\nabla)\mathbf{u}^{(k+1)} + \nabla p^{(k+1)} = \mathbf{f}$$

$$\nabla.\mathbf{u}^{(k+1)} = 0$$

Newton's method

$$\nu\Delta\mathbf{u}^{k+1} + \mathbf{u}^{k+1}.\nabla\mathbf{u}^{k} + \mathbf{u}^{k}.\nabla\mathbf{u}^{k+1} + \nabla p^{k+1} = \mathbf{f} + \mathbf{u}^{k}.\nabla\mathbf{u}^{k},$$

$$\nabla.\mathbf{u}^{k+1} = 0.$$

2. Solution techniques

Matrix form after linearization

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \quad \text{or } \mathcal{A}x = b$$

- $F \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$, $f \in \mathbb{R}^n$ and $m \leq n$
- Sparse linear system, symmetric (Stokes problem), nonsymmetric (Navier Stokes) and always indefinite.
- For unique solution **u** and p, finite elements must satisfy BB condition.
- Saddle point problem having large number of zeros on the main diagonal

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Preconditioner for the Navier Stokes equations

Definition

A linear system $\mathcal{A}x = b$ is transformed into $P^{-1}\mathcal{A}x = P^{-1}b$.

- Eigenvalues of $P^{-1}\mathcal{A}$ are more clustered than $\mathcal{A}$
- $P \approx A$
- $Pz = r$ is cheap to solve for z

Block triangular preconditioners

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BF^{-1} & I \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & F^{-1}B^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ BF^{-1} & I \end{bmatrix} \begin{bmatrix} F & B^T \\ 0 & S \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} F & B^T \\ 0 & S \end{bmatrix}^{-1}, S = -BF^{-1}B^T \text{(Schur complement matrix)}$$

$$Sz_2 = r_2, \quad Fz_1 = r_1 - B^T z_2$$

- GMRES converges in two iterations if exact arithmetic is used [Murphy, Golub, Wathen -2000]
- In practice F^{-1} and S^{-1} are expensive, so they are approximated

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Block triangular preconditioners

- Pressure convection diffusion (PCD) [Kay, Login and Wathen, 2002]

  $S \approx -A_p F_p^{-1} Q_p$

- Least squares commutator (LSC) [Elman, Howle, Shadid, Silvester and Tuminaro, 2002]

  $S \approx -(BQ^{-1}B^T)(BQ^{-1}FQ^{-1}B^T)^{-1}(BQ^{-1}B^T)$

- one of the best approximations available in the literature
- Convergence independent of the mesh size and mildly dependent on Reynolds number
- Require extra operators
- Require iterative solvers (Geometric multigrid, algebraic multigrid) for the (1,1) and (2,2) blocks

Augmented Lagrangian Approach (AL) [Benzi, Olshanski, 2007]

Adapted system

$$\begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

$$\hat{S}^{-1} = -(\nu \hat{Q}_p^{-1} + \gamma W^{-1})$$

- $\hat{Q}_p$ approximation of the pressure mass matrix
- $W = \hat{Q}_p$, γ Lagrange multiplier, ν viscosity
- Convergence independent of the mesh size and mildly dependent on Reynolds number
- Require iterative solvers (Geometric multigrid, algebraic multigrid) for the (1,1) block

Incomplete LU preconditioners

$\mathcal{A} = LU - R$, where R consist of dropped entries that are absent in the index set $\mathcal{S}(i,j)$.

$\mathcal{S} = \{(i,j) | \, a_{ij} \neq 0\}$ [Classical ILU by Meijerink and van der Vorst, 1977].

In our case, $\mathcal{S}(i,j) = \{(i,j) | \; i,j$ are connected in the fi nite element grid$\}$. So zeros in the matrix, due to the coeffi cients are considered to be non-zero in the structure.

- if $||R||$ is large, give poor convergence (**reordering**)
- Instability due to large $||L^{-1}||$ and $||U^{-1}||$

Preconditioners for the Navier Stokes equations

ILUPACK

- Bollhöfer and Saad
- Static reordering schemes
- Inverse-based ILU with diagonal pivoting
- Multilevel framework
- Iterative solver

3. Advanced ILU preconditioner

Effect of reordering

- In direct solver, reordering improves the profi le and bandwidth of the matrix.
- Improve the convergence of the ILU preconditioned Krylov subspace method
- Minimizes dropped entries in ILU ($\|\mathcal{A} - \bar{L}\bar{U}\|_F$)
- May give stable factorization ($\|I - \mathcal{A}(\bar{L}\bar{U})^{-1}\|_F$)

[Dutto-1993, Benzi-1997, Duff and Meurant-1989, Wille-2004, Chow and Saad - 1997]

Well-known renumbering schemes

- Cuthill McKee renumbering (RCM) [Cuthill McKee - 1969]
- Sloan renumbering [Sloan - 1986]
- Minimum degree renumbering (MD) [Tinney and Walker - 1967]

New renumbering scheme

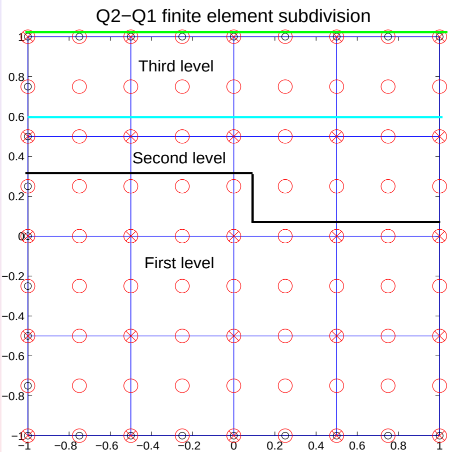
- Renumbering of grid points: Grid points are renumbered with Sloan or Cuthill McKee algorithms
- The unknowns are reordered by p-last or p-last per level methods

In **p-last reordering**, first all the velocity unknowns are ordered followed by pressure unknowns. Usually, this produces a large profile but avoids breakdown of the LU decomposition.

p-last per level reordering, smaller profile

p-last per element reordering, smallest profile

p-last per level reordering

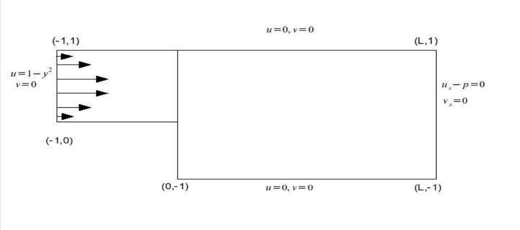


Special features of Advanced ILU

- lumping of positive off-diagonal elements
- extra fill in (global, or pressure only)
- ϵ stabilization parameter

Flow domains

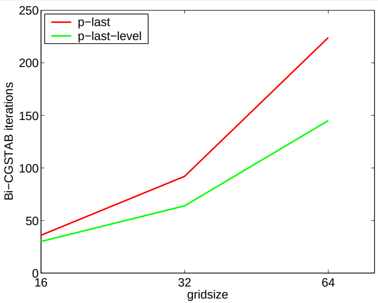
- **Channel flow** The Poiseuille channel flow in a square domain $(-1,1)^2$ with a parabolic inflow boundary condition and the natural outflow condition having the analytic solution: $u_x = 1 - y^2$; $u_y = 0$; $p = 2\nu x$
- **Backward facing step**

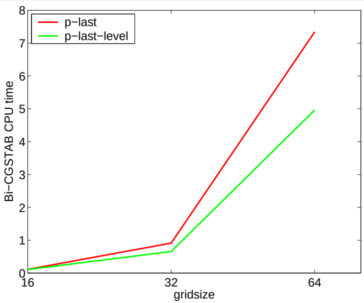


- Q2-Q1 fi nite element discretization [Taylor, Hood - 1973]
- Q2-P1 fi nite element discretization [Crouzeix, Raviart - 1973]

Comparison of p-last and p-last per level

Square channel, Stokes, Q2-P1



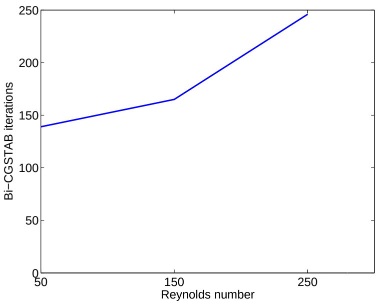


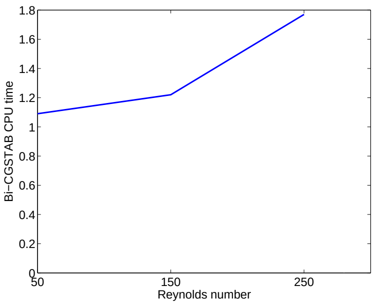
GMRES(20) costs more CPU time

GMRESR is comparable with Bi-CGSTAB, wrt CPU time

Dependence on the Reynoldsnumber

Backward Facing Step, Navier Stokes, 16 $\times$ 48 with Q2-Q1 discretization





Sloan reordering is faster than RCM reordering

Advanced ILU compared with PCD and LSC

Comparison of ILU preconditioner with block triangular preconditioners using GMRES (*accuracy* $= 10^{-4}$) and Newton linearization for the backward facing step Navier Stokes problem, Q2-Q1 element

Direct solver for (1,1),(2,2) blocks of the block triangluar preconditioners

Re=100	PCD		p-last-level(Sloan)		LSC	
-	Iter	sec	Iter	sec	Iter	sec
8x24	32	0.50	16	0.03	15	0.33
16x24	33	1.21	21	0.08	15	0.76
32x24	37	3.16	68	0.67	18	2.10
64x24	45	8.30	61	1.13	25	9.10
Re = 200						
8x24	45	7.26	60	0.10	23	0.50
16x24	50	1.90	37	0.15	22	1.14
32x24	52	4.30	83	0.75	24	2.73
64x24	60	11.0	41	0.80	29	7.00

Advanced ILU compared with AL and LSC

Comparison of ILU preconditioner with block triangular preconditioners using GCR(30) and Newton linearization for the backward facing step Navier Stokes problem

Iterative solver for (1,1),(2,2) blocks of the block triangluar preconditioners

Q2-Q1	AL		LSC		p-last-level(Sloan)	
-	Iter	sec	Iter	sec	Iter	sec
8x24	9	0.10	17	0.11	17	0.02
16x24	9	0.44	18	0.20	23	0.06
32x24	9	2.72	23	0.48	58	0.25
64x24	9	9.30	27	1.20	59	0.56
128x24	9	44.5	42	3.90	488	11.0

Q2-P1	AL		LSC		p-last-level(Sloan)	
8x24	8	0.12	14	0.14	84	0.11
16x24	8	0.28	11	0.24	118	0.32
32x24	8	0.64	20	1.00	220	1.20
64x24	8	1.50	NC		308	3.50
128x24	8	3.43	NC		NC	

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ILUPACK with GMRES(20)

Grid	Iterations	nnz(A)	nnz(ILU)	Growth factor
8x24	4	7040	15020	2.13
16x48	4	33122	96227	2.90
32x96	4	143548	797598	5.56
64x192	5	598832	3951127	6.60

Backward facing step, Stokes problem, Q2-Q1

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ILUPACK with GMRES(20)

	ILUPACK		ILU p-last-level(Sloan)	
Grid	Iter. (time(s))	Total time(s)	Iter. (time(s))	Total time(s))
8x24	4 (0.01)	0.04	14(0.008)	0.03
16x48	4(0.04)	0.23	20(0.012)	0.07
32x96	4 (0.19)	2.42	87(0.30)	0.41
64x192	5 (1.0)	21.00	276(3.35)	3.92

Backward facing step, Stokes problem, Q2-Q1

5. Conclusions

- IFISS is a nice tool to investigate the incompressible Navier Stokes equations
- Advanced ILU:
  - renumbering of grid points and reordering of unknowns
  - no break down and fast convergence
  - iterations increase with increase in Reynolds number and grid points
- Block preconditioners are better for large grid sizes and large Reynolds numbers
- ILUPACK needs small number of iterations, but memory and CPU time can be large
- Stretched grids?

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