Deflation acceleration of the PCG method applied to porous media flow

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SIAM Conference on Mathematical and Computational Issues in the Geosciences

Domain-based and Krylov-based Deflation Methods for Flow in Porous Media

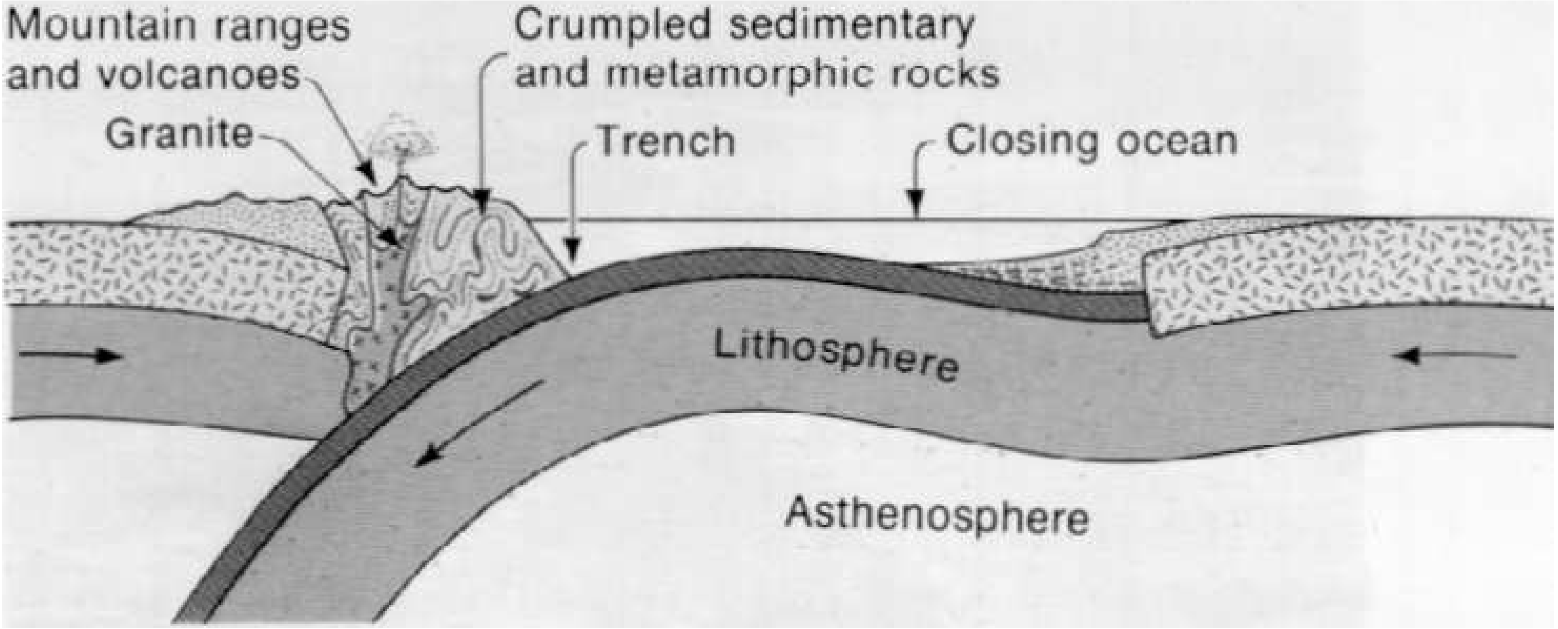
March 20, 2007, Santa Fé, New Mexico

Contents

1. Introduction

Motivation

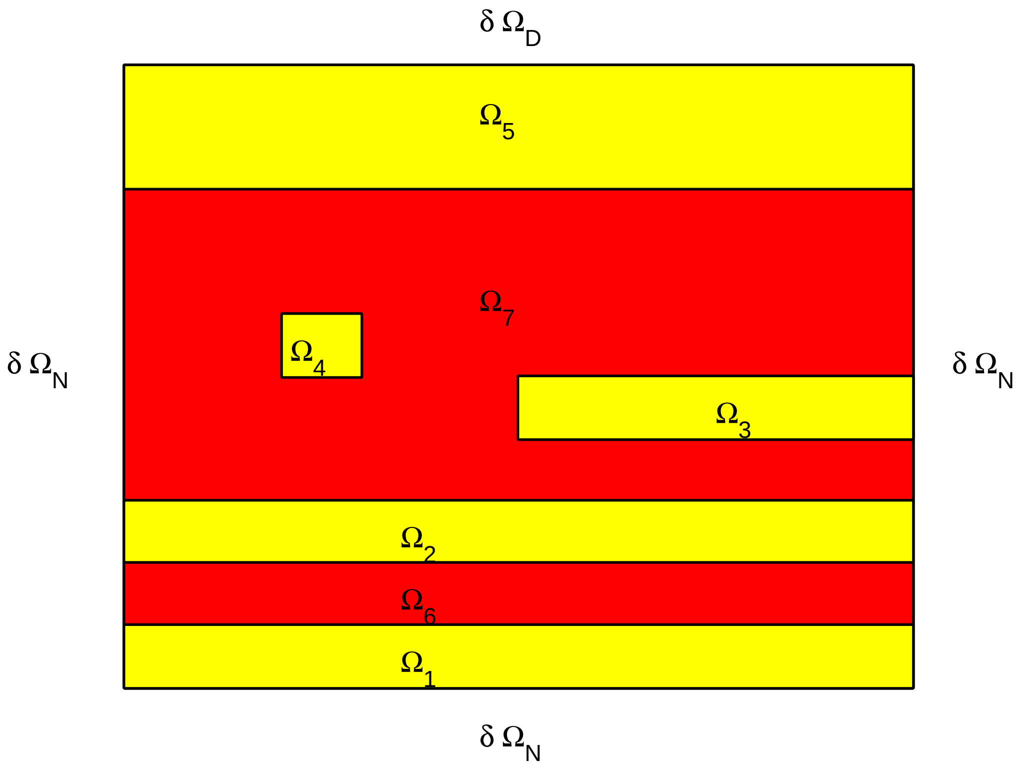
Knowledge of the fluid pressure in rock layers is important for an oil company to predict the presence of oil and gas in reservoirs.



The earth's crust has a layered structure

Mathematical model

Computation of fluid pressure $-\text{div}(\sigma \nabla p(x)) = 0$ on Ω, p fluid pressure, σ permeability



$\sigma_h = 1$ (sand) $\sigma_l = \varepsilon = 10^{-7}$ (shale)

Properties and Applications

$$Ax = b$$

A is sparse and SPD

Condition number of A is $O(10^7)$, due to large contrast in permeability

Applications

- reservoir simulations
- porous media flow
- electrical power networks
- semiconductors
- magnetic field simulations

- fictitious domain methods

2. IC preconditioned CG

Error estimate

$$Ax = b$$

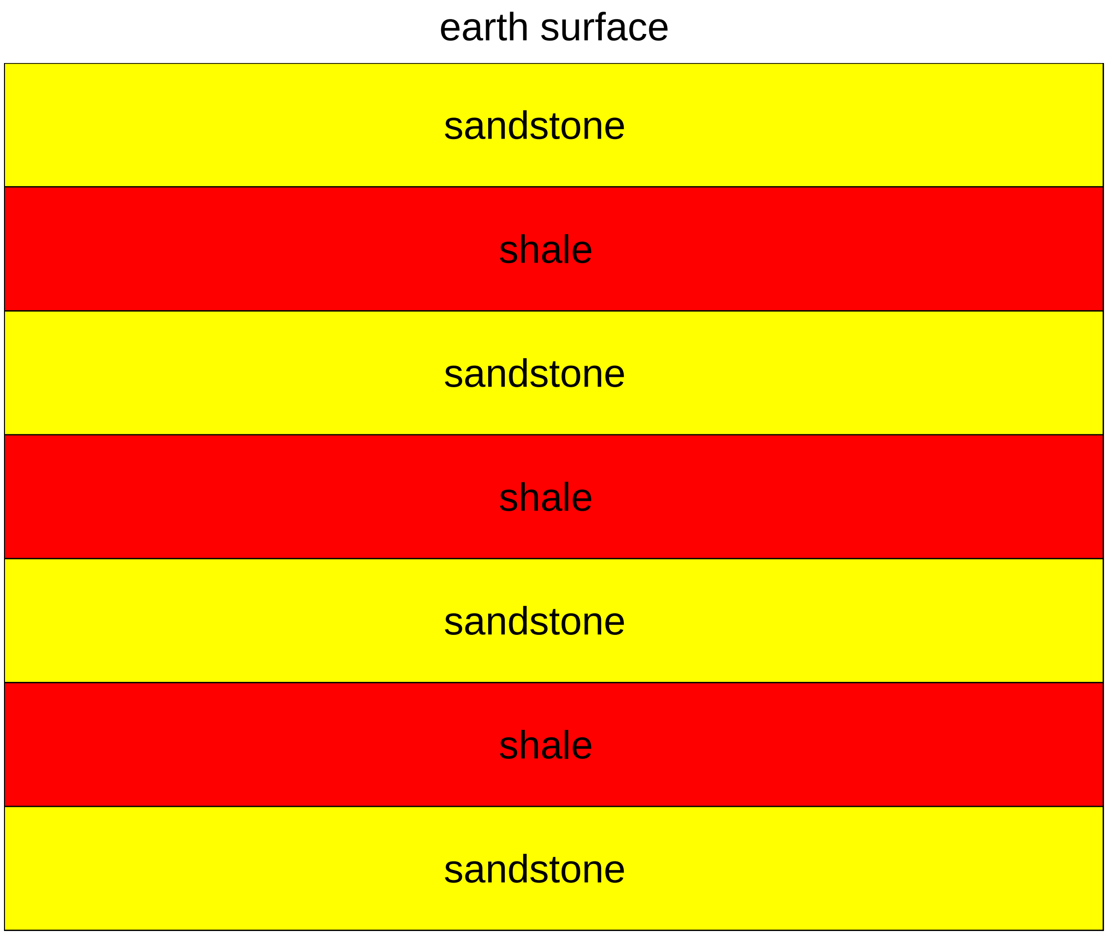
$$M^{-1}Ax = M^{-1}b$$

$$x - x_k = (M^{-1}A)^{-1}M^{-1}A(x - x_k)$$

$$\|x - x_k\|_2 \leq \frac{1}{\lambda_{min}} \|M^{-1}r_k\|_2$$

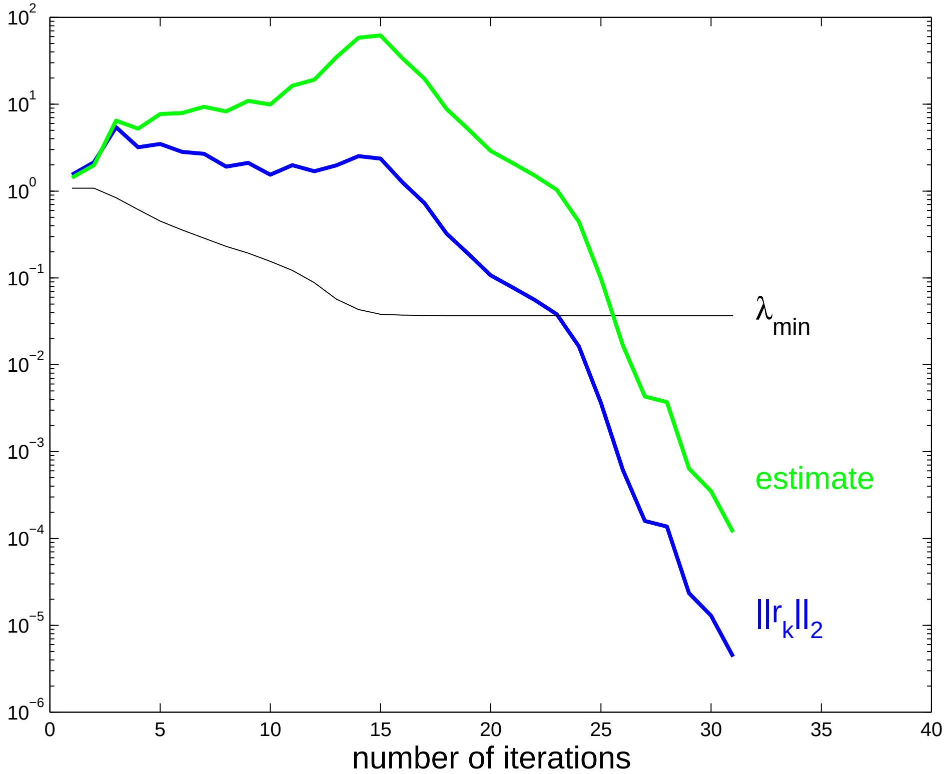
λ_{min}: smallest eigenvalue of $M^{-1}A$

Test problem



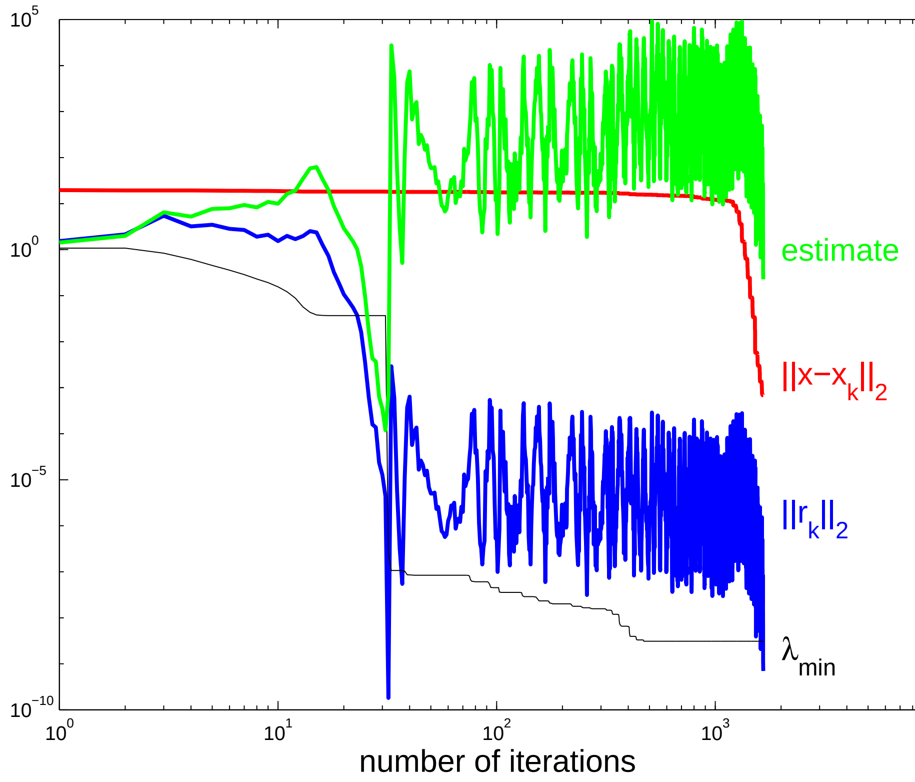
Configuration with 7 straight layers

Convergence CG



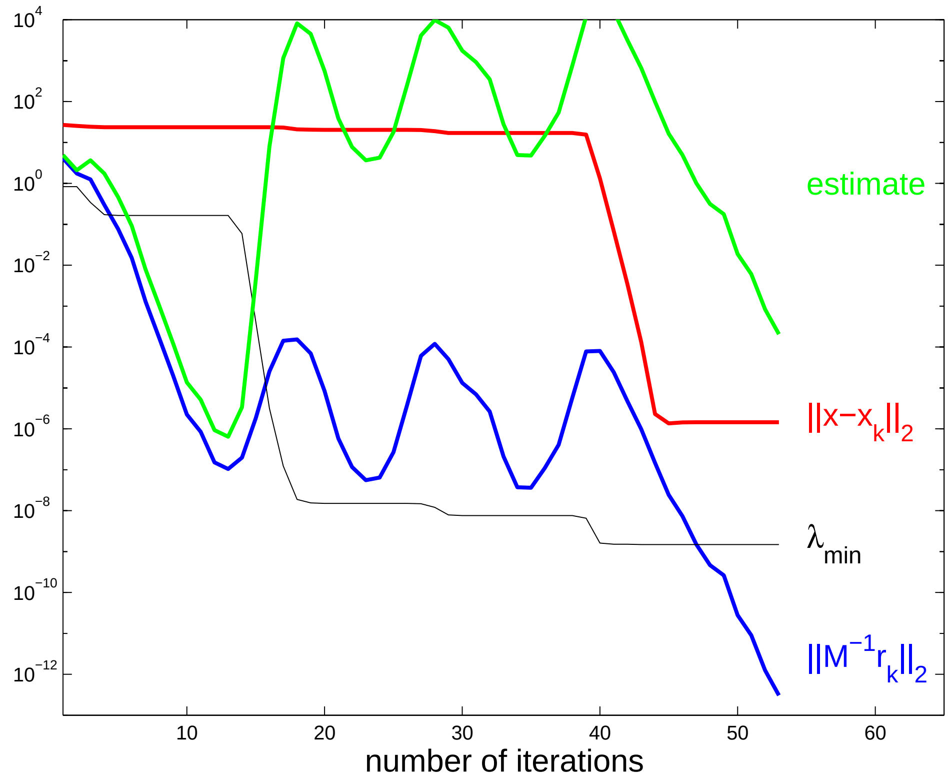
Convergence behavior of CG without preconditioning

Convergence CG



Convergence behavior of CG without preconditioning

Convergence ICCG



Convergence behavior of ICCG

Spectrum of IC preconditioned matrix

L is the Incomplete Cholesky factor of A

k^s is the number of high-permeability domains not connected to a Dirichlet boundary

D is a diagonal matrix $(d_{ii} > 0)$ and $\hat{A} = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$

Theorem 1 (scaling invariance)
$L^{-1} A L^{-T}$ and $\hat{L}^{-1} \hat{A} \hat{L}^{-T}$ are identical.

Proof:

$\hat{L} = D^{-\frac{1}{2}} L$ and $\hat{L}^{-1} \hat{A} \hat{L}^{-T} = L^{-1} D^{\frac{1}{2}} (D^{-\frac{1}{2}} A D^{-\frac{1}{2}}) D^{\frac{1}{2}} L^{-T} = L^{-1} A L^{-T}$.

Spectrum of IC preconditioned matrix

Take D =diag(A)

Theorem 2

$\hat{A}$ has k^s eigenvalues of $O(\varepsilon)$, where ε is the ratio between high and low permeability.

Theorem 3

The IC preconditioned matrix $L^{-1}AL^{-T}$ has k^s eigenvalues of $O(\varepsilon)$.

Proof: Scaling invariance (Theorem 1) implies

$$\text{spectrum}(L^{-1}AL^{-T}) = \text{spectrum}(\hat{L}^{-1}\hat{A}\hat{L}^{-T})$$

In [Vuik, Segal, Meijerink, Wijma, 2001] we have shown that the number and size of small eigenvalues of $\hat{A}$ and $\hat{L}^{-1}\hat{A}\hat{L}^{-T}$ are the same. The theorem is proven by using Theorem 2. ⊠

3. Deflated ICCG

Idea: remove the bad eigenvectors from the error/residual.

Krylov	Ar
Preconditioned Krylov	$M^{-1}Ar$
Block Preconditioned Krylov	$\sum_{i=1}^{m}(M_i^{-1})Ar$
Block Preconditioned Deflated Krylov	$\sum_{i=1}^{m}(M_i^{-1})PAr$

3. Deflated ICCG

Idea: remove the bad eigenvectors from the error/residual.

Various choices are possible:

- Projection vectors
  Physical vectors, eigenvectors, coarse grid projection vectors (constant, linear, ...)
- Projection method
  Deflation, coarse grid projection, balancing, augmented, FETI
- Implementation
  sparseness, with(out) using projection properties, optimized, ...

Literature

Deflated CG (start)

Nicolaides 1987, Mansfield 1990

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Deflated CG (further development)

Graham and Hagger 1997, 1999, Kolotilina 1998, Vuik, Segal and Meijerink 1999, Saad, Yeung, Erhel and Guyomarc'h 2000, Frank and Vuik 2001, Aksoylu, Rodriguez, Klie and Wheeler 2006, Nabben and Vuik 2004, 2006, Scheichl and Graham, 2006

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Deflation and restarted GMRES

Morgan 1995, Erhel, Burrage and Pohl 1996, Chapman and Saad 1997, Morgan 2002, Giraud and Gratton 2005, Kilmer and De Sturler 2006

Deflated ICCG

A is SPD, Conjugate Gradients

$$P = I - AZE^{-1}Z^T \text{ with } E = Z^TAZ$$

and $Z = [z_1 ... z_m]$, where $z_1, ..., z_m$ are independent deflation vectors.

Properties

1. $P^TZ = 0$ and $PAZ = 0$
2. $P^2 = P$
3. $AP^T = PA$

Deflated ICCG

$$x = (I - P^T)x + P^T x$$

$$(I - P^T)x = ZE^{-1}Z^T Ax = ZE^{-1}Z^T b, \qquad AP^T x = PAx = Pb$$

Deflated ICCG

$$x = (I - P^T)x + P^T x$$

$$(I - P^T)x = ZE^{-1}Z^T Ax = ZE^{-1}Z^T b, \qquad AP^T x = PAx = Pb$$

DICCG

$k = 0,\ \hat{r}_0 = Pr_0,\ p_1 = z_1 = L^{-T}L^{-1}\hat{r}_0;$

while $\|\hat{r}_k\|_2 > \varepsilon$ **do**

$k = k + 1;$

$\alpha_k = \frac{(\hat{r}_{k-1}, z_{k-1})}{(p_k, PAp_k)};$

$x_k = x_{k-1} + \alpha_k p_k;$

$\hat{r}_k = \hat{r}_{k-1} - \alpha_k PAp_k;$

$z_k = L^{-T}L^{-1}\hat{r}_k;$

$\beta_k = \frac{(\hat{r}_k, z_k)}{(\hat{r}_{k-1}, z_{k-1})}; \qquad p_{k+1} = z_k + \beta_k p_k;$

end while

Convergence and termination criterion

Choose z_1 , z_2 , z_3 eigenvectors of $L^{-T}L^{-1}A$

Convergence

$$\|P^Tx - P^Tx_k\|_2 \leq 2\sqrt{K}\|P^Tx - P^Tx_0\|_2 \left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^k$$

where $K = \frac{\lambda_n}{\lambda_4}$

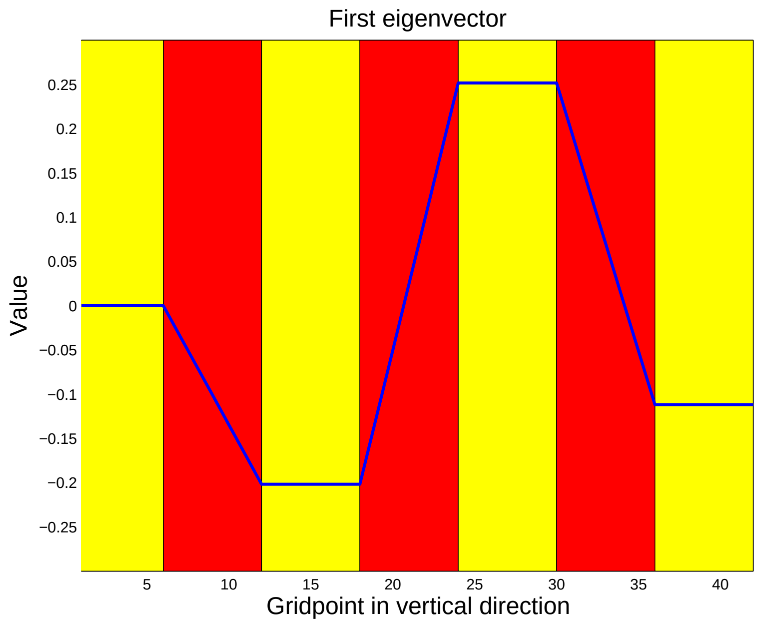
Termination criterion

$$\|L^{-T}L^{-1}Pb - L^{-T}L^{-1}PAx_k\|_2 \leq \frac{\delta}{\lambda_4} \text{ implies } \|P^Tx - P^Tx_k\|_2 \leq \delta$$

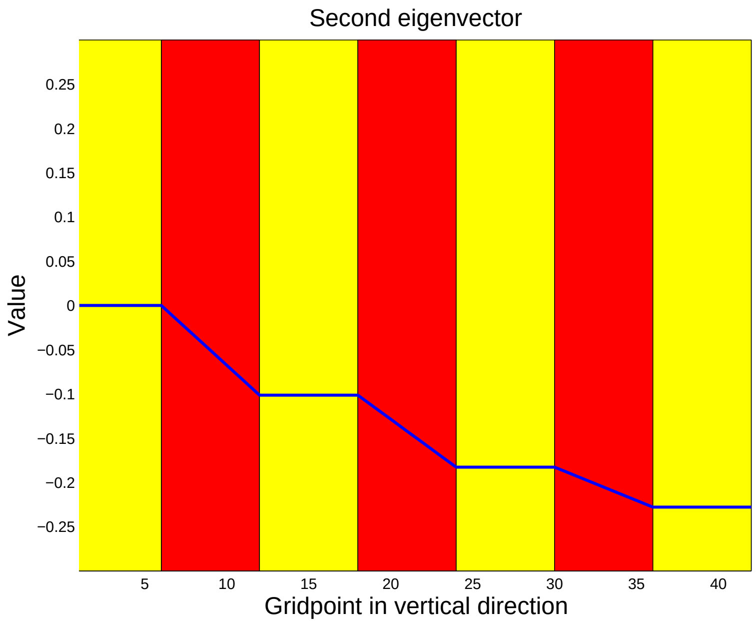
Deflation vectors

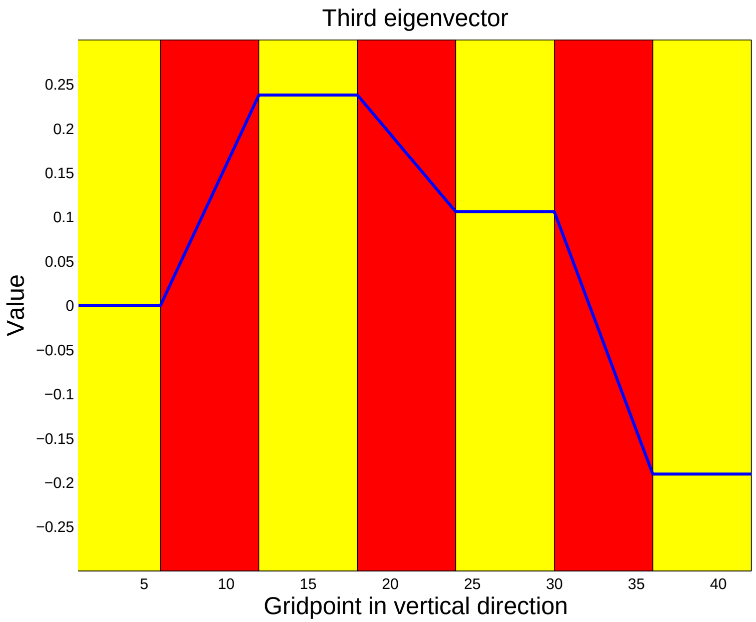
Choose eigenvectors of $L^{-T}L^{-1}A$. Properties of cross sections:

- a constant value in sandstone layers
- in shale layers their graph is linear



Eigenvectors of $L^{-T}L^{-1}A$





4. Physical deflation vectors

k is number of subdomains

$\Omega_i, i = 1, ..., k^s$ high-permeability subdomains without a Dirichlet B.C.;
$i = k^s + 1, ..., k^h$ remaining high-permeability subdomains

- define z_i for $i \in \{1, ..., k^s\}$
- $z_i = 1$ on $\bar{\Omega}_i$ and $z_i = 0$ on $\bar{\Omega}_j, j \neq i, j \in \{1, ..., k^h\}$
- z_i satisfies equation:

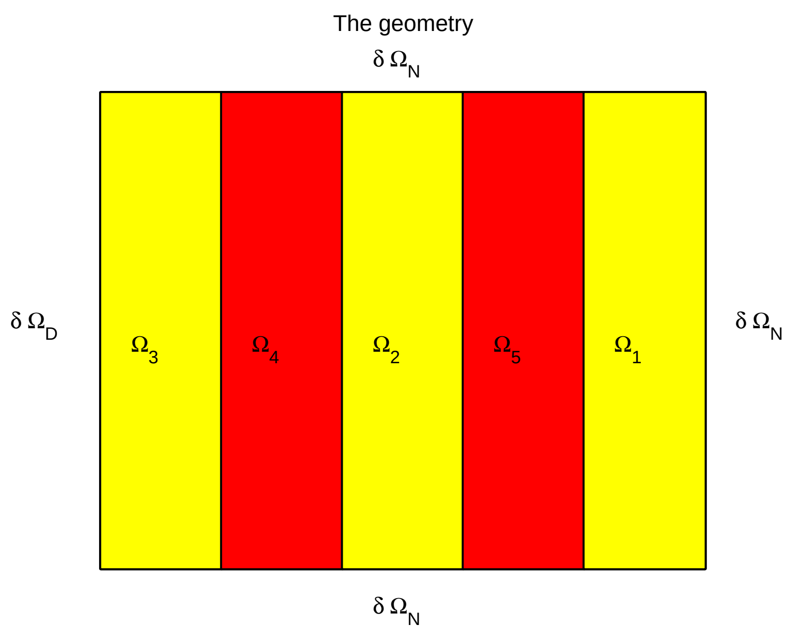
$$-\text{div}(\sigma_j \nabla z_i) = 0 \text{ on } \Omega_j, j \in \{k^h + 1, ..., k\},$$

with appropriate boundary conditions

Sparse vectors, subproblems are cheap to solve

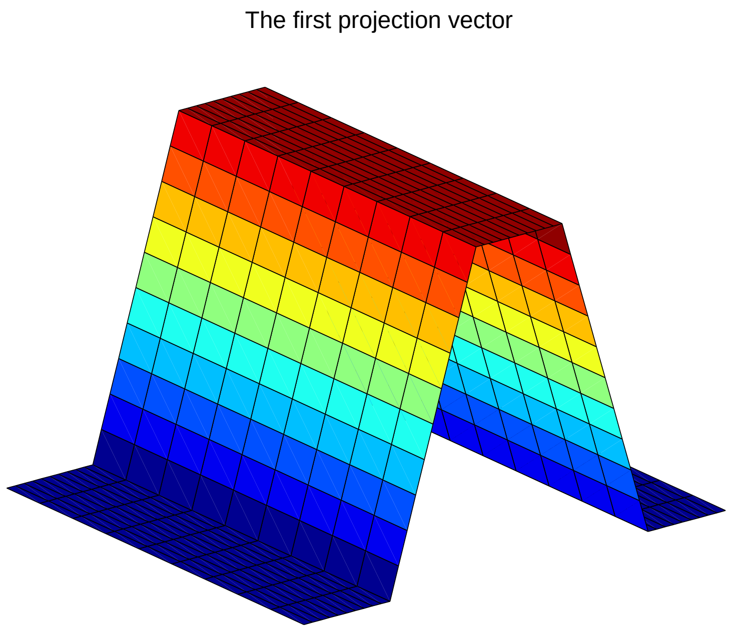
Physical deflation vectors

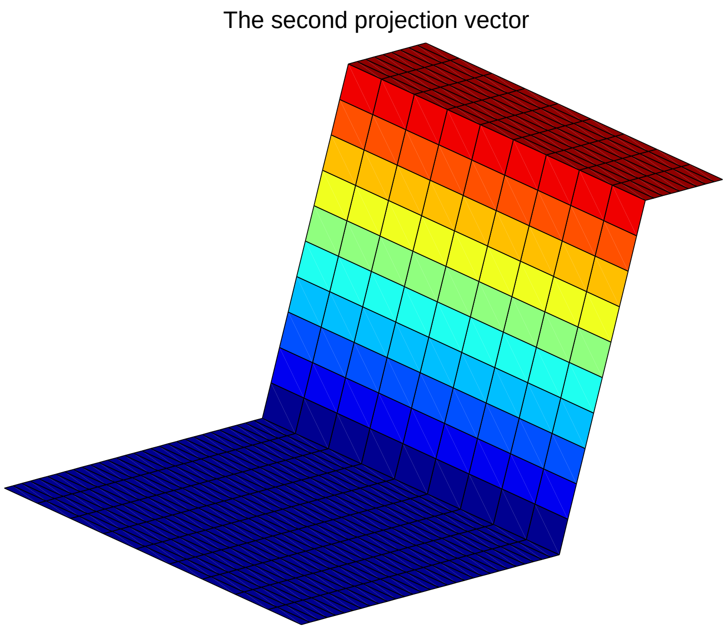
Example with $k_s = 2, k_h = 3$, and $k = 5$



Physical deflation vectors

Example with $k_s = 2, k_h = 3$, and $k = 5$





Properties

Theorem 4

The deflation vectors are such that for $D = \text{diag}(A)$

- $\|D^{-1}Az_i\|_\infty = O(\varepsilon)$
- $\|L^{-T}L^{-1}Az_i\|_2 = O(\varepsilon)$

Define $Z = [z_1 ... z_{k^s}]$ and $U = [u_1 ... u_{k^s}]$, where u_i are 'small' eigenvectors.

Theorem 5

There is a matrix X such that $Z = UX + E$, with $\|E\|_2 = O(\sqrt{\varepsilon})$

Sensitivity of deflation vectors

- Random vector added in shale layers (amplitude $\alpha/2$)

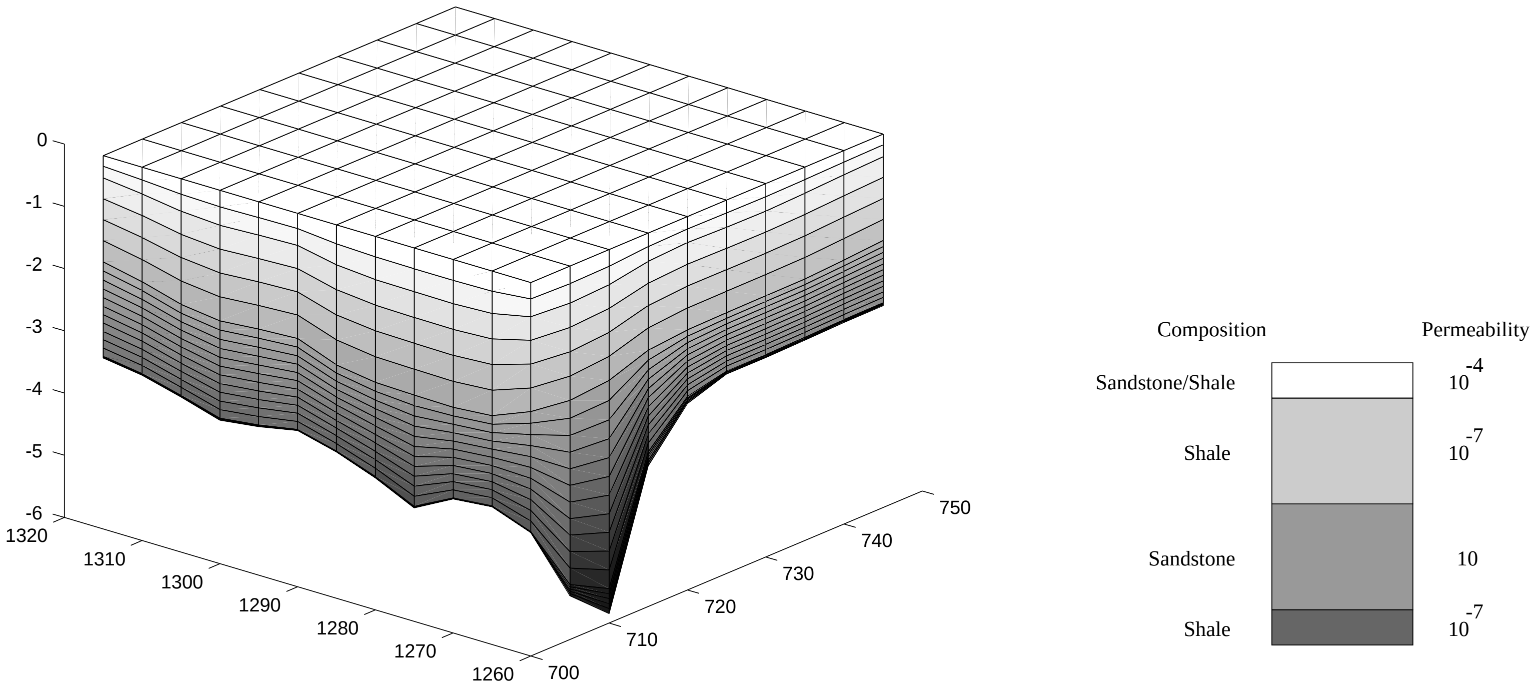
α	0	10^{-1}	1	ICCG
λ_{per}	0.164	0.164	$8.2 \cdot 10^{-3}$	$1.6 \cdot 10^{-9}$
iter	14	15	24	54

- Random vector added to the nonzero parts

α	0	10^{-3}	10^{-1}	ICCG
λ_{per}	0.164	$9 \cdot 10^{-4}$	$9 \cdot 10^{-8}$	$1.6 \cdot 10^{-9}$
iter	14	27	56	54

After perturbation the smallest eigenvalues remain exactly zero, however, the smallest non-zero eigenvalue can change considerably.

Geometry oil flow problem



Results oil flow problem

Varying σ_{shale}

σ	ICCG		DICCG	
	λ_{min}	iter	λ_{min}	iter
10^{-3}	$1.5 \cdot 10^{-2}$	26	$6.9 \cdot 10^{-2}$	20
10^{-5}	$2.2 \cdot 10^{-4}$	59	$7.7 \cdot 10^{-2}$	20
10^{-7}	$2.3 \cdot 10^{-6}$	82	$7.7 \cdot 10^{-2}$	20

Varying accuracy

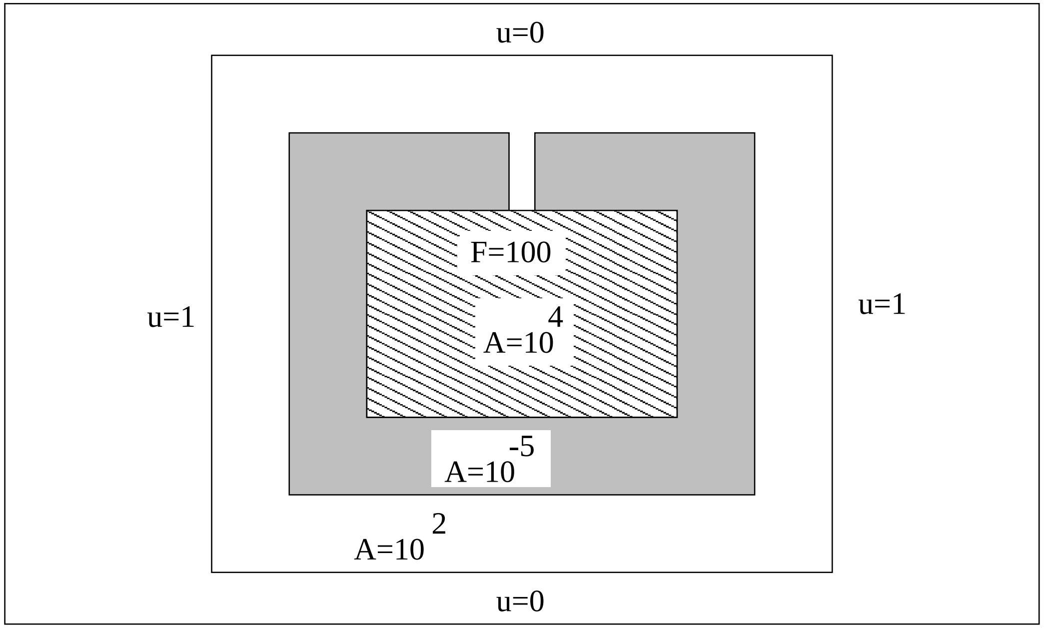
accuracy	ICCG		DICCG	
	iter	CPU	iter	CPU
10^{-5}	82	18.9	20	6.3
10^{-3}	78	18.0	12	4.1
10^{-1}	75	17.2	2	1.2

A groundwater flow problem

The pressure in groundwater satisfies the equation:

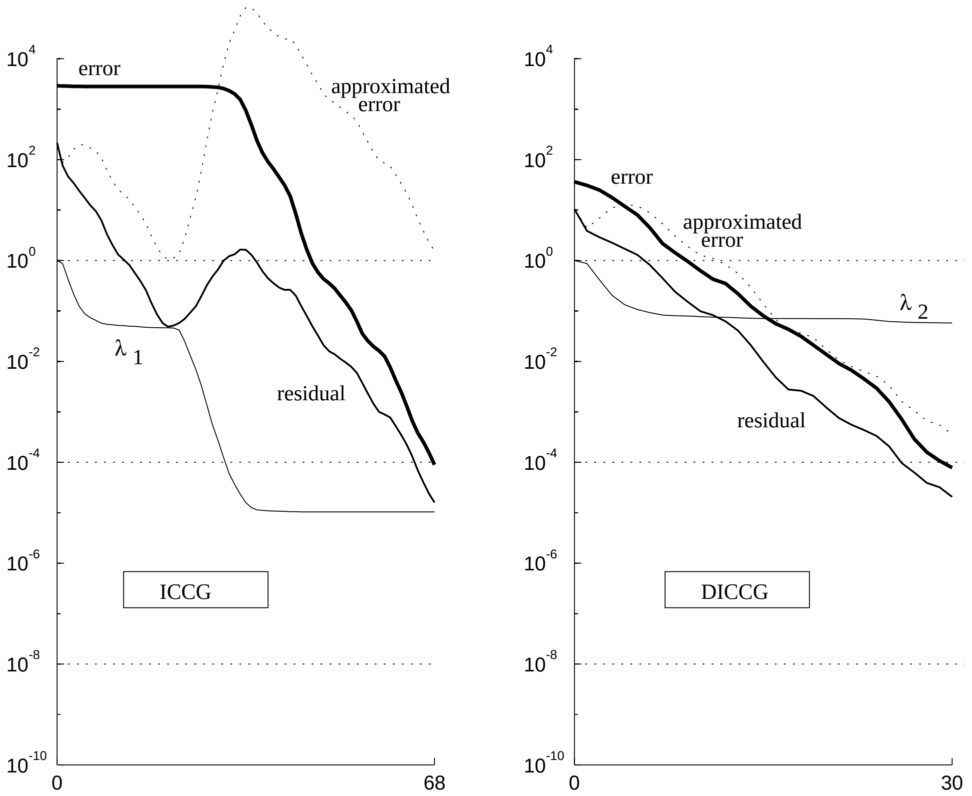
$$-\nabla \cdot (A \nabla u) = F, \tag{1}$$

where the coefficients and geometry of the problem are:



A groundwater flow problem

The low permeable layer ($A = 10^{-5}$) and the jump in permeabilities between the two sand sections lead to a 'small' eigenvalue.



5. Conclusions

- DICCG is a robust and efficient method to solve diffusion problems with discontinuous coefficients.
- The choice of the projection vectors is important for the success of a projection method.
- For layered problems the physical deflation vectors are the optimal choice for the projection vectors.
- For many problems a second level preconditioner (Deflation) saves a lot of CPU time.

Further information

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