The Construction of Projection Vectors for a Deflated ICCG Method used in Problems with a Layered Structure

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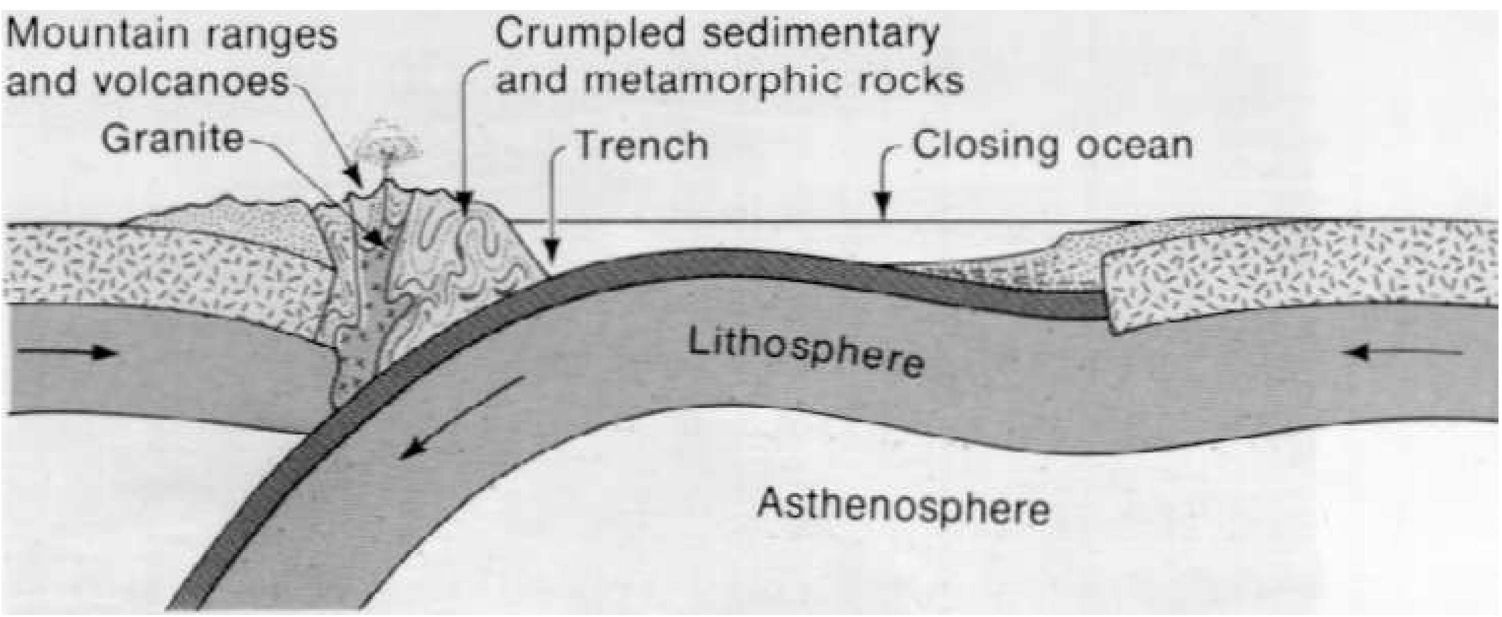

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Statement of the problem

Motivation

Knowledge of the fluid pressure in rock layers is important for an oil company to predict the presence of oil and gas in reservoirs.

Mathematical model

The fluid pressures are described by a time-dependent 3 dimensional diffusion equation, based on conservation of mass and Darcy's law.

Numerical method

The Finite Element Method and the Euler Backward method are used. In each time-step a large, sparse linear system has to be solved.

Iterative solution method

A preconditioned Conjugate Gradient method is used. Due to large contrasts in permeability the extreme eigenvalues differ orders of magnitude. This leads to slow convergence of ICCG and conventional termination criteria are no longer reliable.

Deflated ICCG

The projection P is defined as

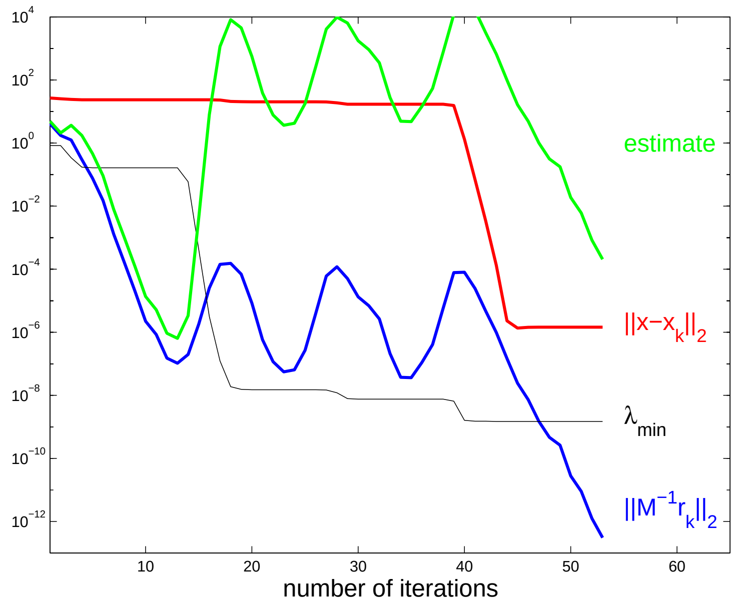
$P = I - VE^{-1}(AV)^T$ with $E = (AV)^T V$ and $V = [v_1 ... v_m]$,

where $v_1, ..., v_n$ are the independent projection vectors.

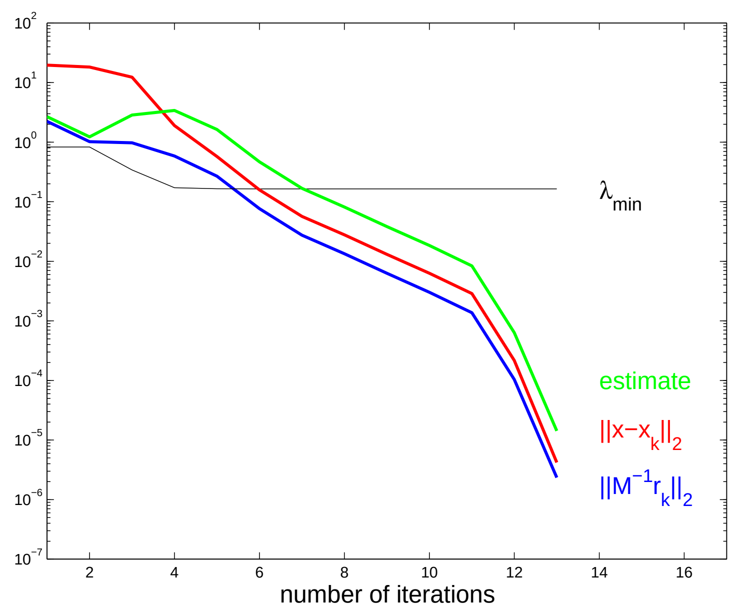
DICCG

$k = 0,\ \hat{r}_0 = P^T r_0,\ p_1 = z_1 = L^{-T} L^{-1} \hat{r}_0;$
while $\|\hat{r}_k\|_2 > \varepsilon$ **do**
$\quad k = k + 1;$
$\quad \alpha_k = \frac{(\hat{r}_{k-1}, z_{k-1})}{(p_k, P^T A p_k)};$
$\quad x_k = x_{k-1} + \alpha_k p_k;$
$\quad \hat{r}_k = \hat{r}_{k-1} - \alpha_k P^T A p_k;$
$\quad z_k = L^{-T} L^{-1} \hat{r}_k;$
$\quad \beta_k = \frac{(\hat{r}_k, z_k)}{(\hat{r}_{k-1}, z_{k-1})};$
$\quad p_{k+1} = z_k + \beta_k p_k;$
end while

Results of ICCG



Results of DICCG



Construction of the projection vectors (1)

$$-\mathrm{div}(\sigma \nabla p) = 0 \text{ in } \Omega, \qquad (1)$$

$$p = f \text{ on } \partial\Omega_D, \frac{\partial p}{\partial n} = g \text{ on } \partial\Omega_N \ (\partial\Omega = \partial\Omega_D \cup \partial\Omega_N).$$

Properties of σ are:

$$\sigma(x) = \sigma_h \ \ (= 1 \text{ sandstone }), \ x \in \Omega_i, \in \{1, ..., k_h\},$$

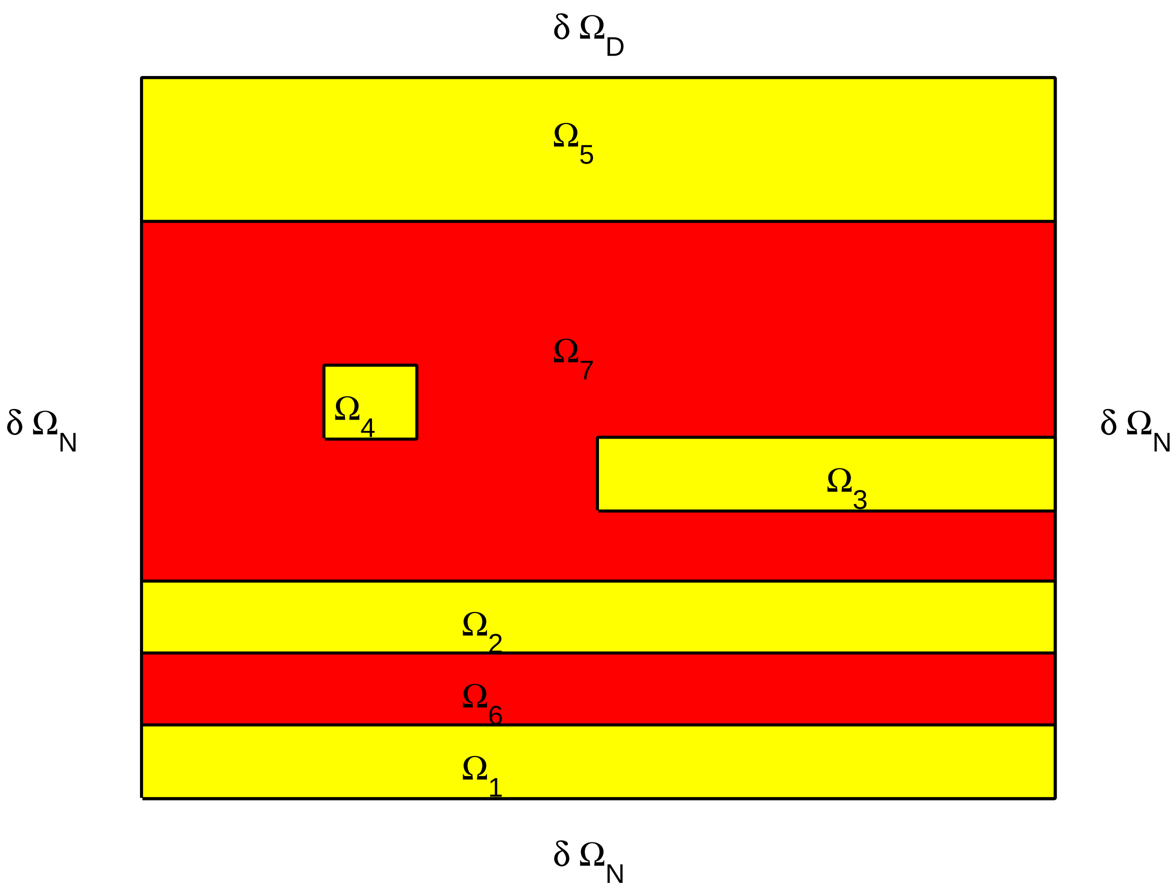
$$\sigma(x) = \sigma_l \ \ (= 10^{-7} \text{ shale }), \ x \in \Omega_i, \in \{k_h + 1, ..., k\},$$

where $\cup_{i=1}^{k} \bar{\Omega}_i = \bar{\Omega}$ and when $\bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset$ then $\sigma_i \neq \sigma_j$

and $\bar{\Omega}_i \cap \partial\Omega_D = \emptyset, \ i \in \{1, ..., k_s\}$.

Construction of the projection vectors (2)

Example with $k_s = 4, k_h = 5$, and $k = 7$



Construction of the projection vectors (3)

Number of small eigenvalues

The spectrum of the IC preconditioned matrix contains k_s small eigenvalues (O(10^{-7}) for shale/sandstone layers).

The projection vectors

The 'small' eigenspace is approximated by the span of the vectors v_i:

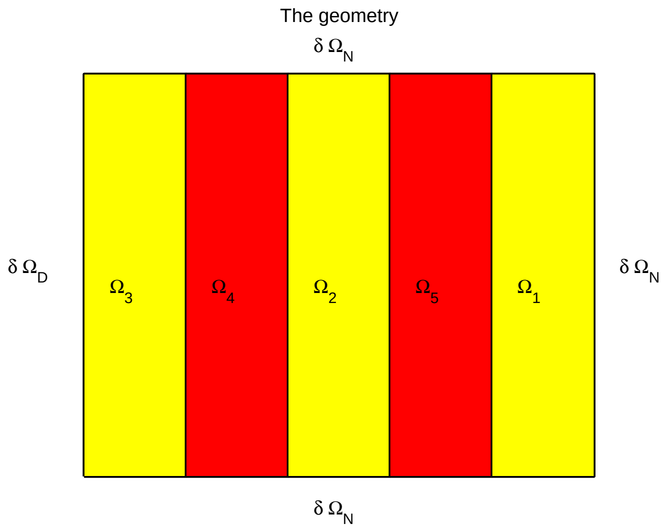
- $(v_i)_m = 1, x_m \in \Omega_i, i \in \{1, ..., k_s\}$,
- $(v_i)_m = 0, x_m \in \Omega_j, j \neq i, j \in \{1, ..., k_h\}$,
- v_i satisfies (1) in the low permeable layers.

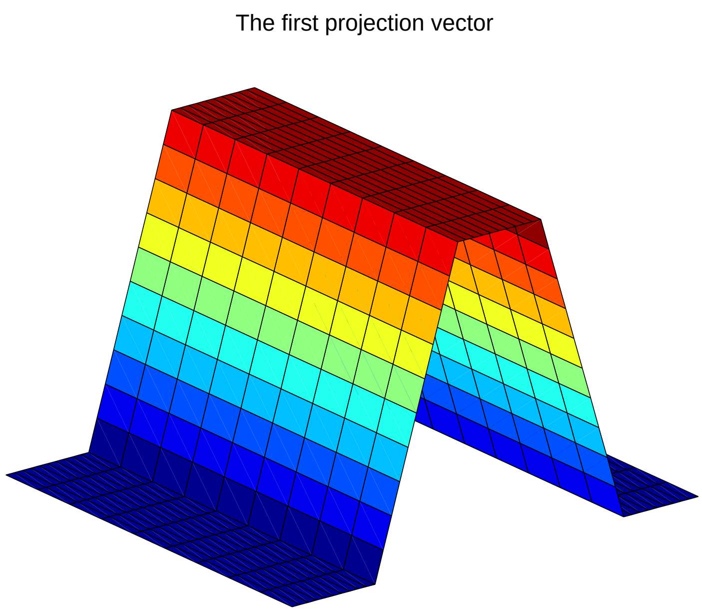
Properties

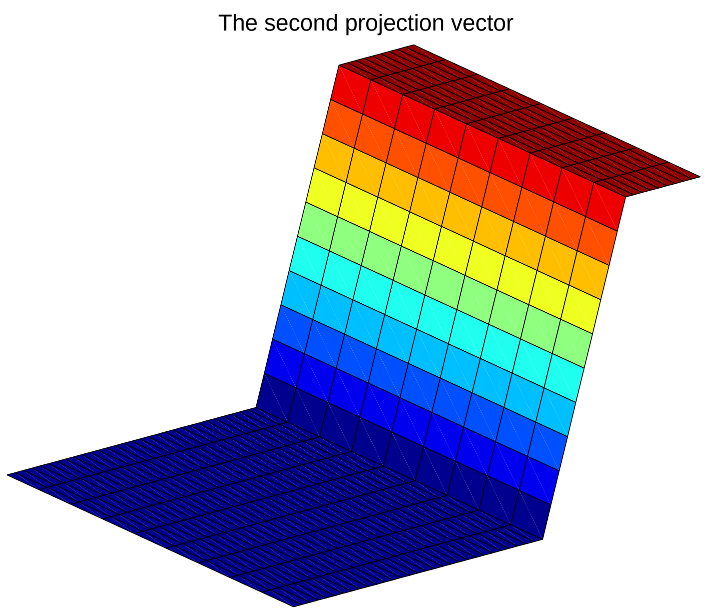
- The vectors v_i and Av_i are sparse (N_i non-zero elements).
- Extra work per iteration: $3 \sum_{i=1}^{k_s} N_i$
- Extra memory: $2 \sum_{i=1}^{k_s} N_i$

Construction of the projection vectors (4)

Example with $k_s = 2$, $k_h = 3$, and $k = 5$







Sensitivity of DICCG (1)

High permeable layers

DICCG is sensitive with respect to perturbations of the projection vectors in the high permeable layers.

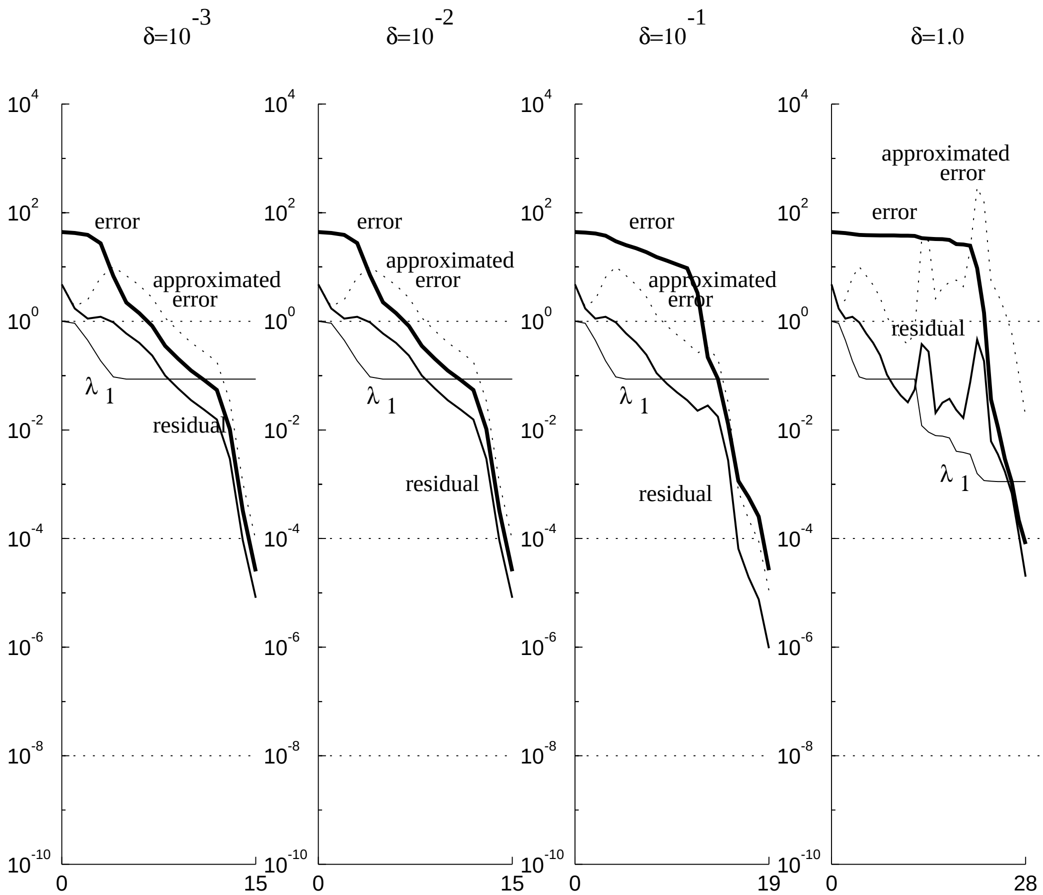
Low permeable layers

The sensitivity is investigated by numerical experiments. As a test example we consider a two-dimensional problem with 4 horizontal sandstone layers separated by 3 horizontal shale layers. We add a random vector with components less than $\delta/2$, to the non-constant parts of the projection vectors.

δ	0	10^{-3}	10^{-2}	10^{-1}	1	ICCG
n	15	15	15	19	28	58

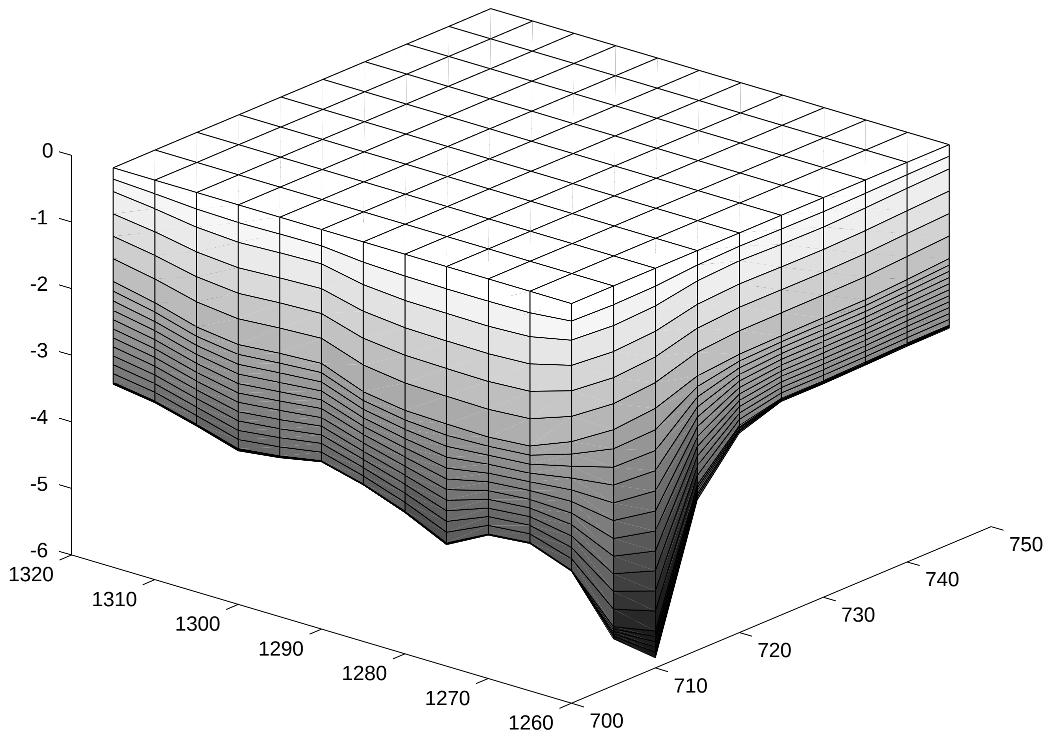
Number of iterations (n) before (D)ICCG reaches the required accuracy (10^{-4})

Sensitivity of DICCG (2)

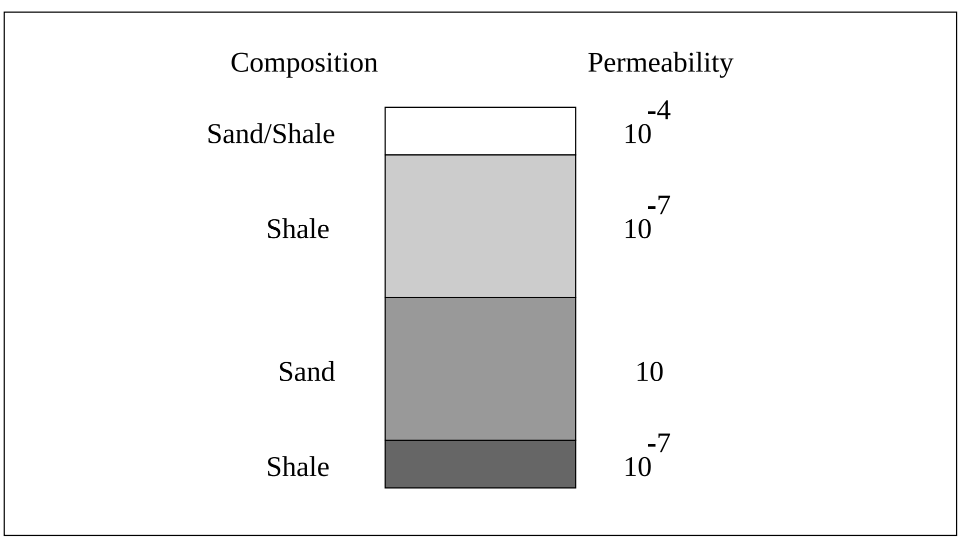


Convergence behavior of DICCG for various values of δ

An oil flow problem (1)



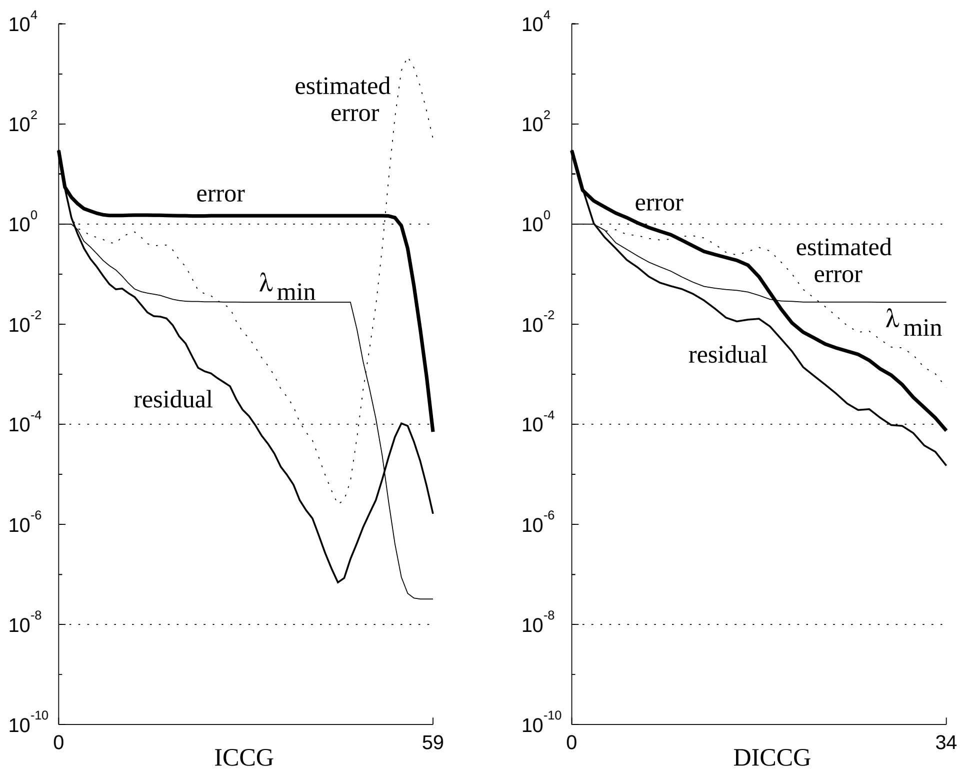
A realistic test structure from Shell



Permeabilities for each layer

An oil flow problem (2)

A 3D problem proposed by Shell consists of four layers. The permeabilities in each layer are chosen constant. There is only one free sand layer, hence there is only one 'small' eigenvalue. The convergence of ICCG and DICCG is given below.

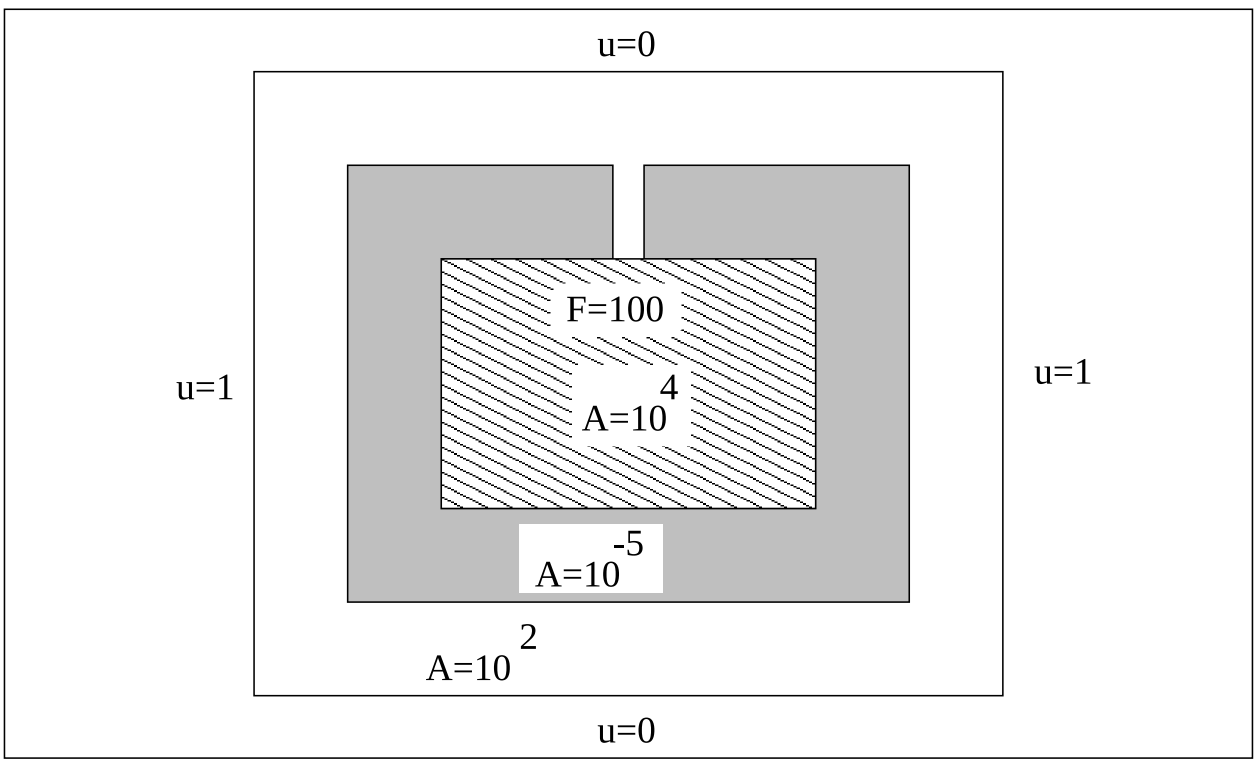


A groundwater flow problem (1)

The pressure in groundwater satisfies the equation:

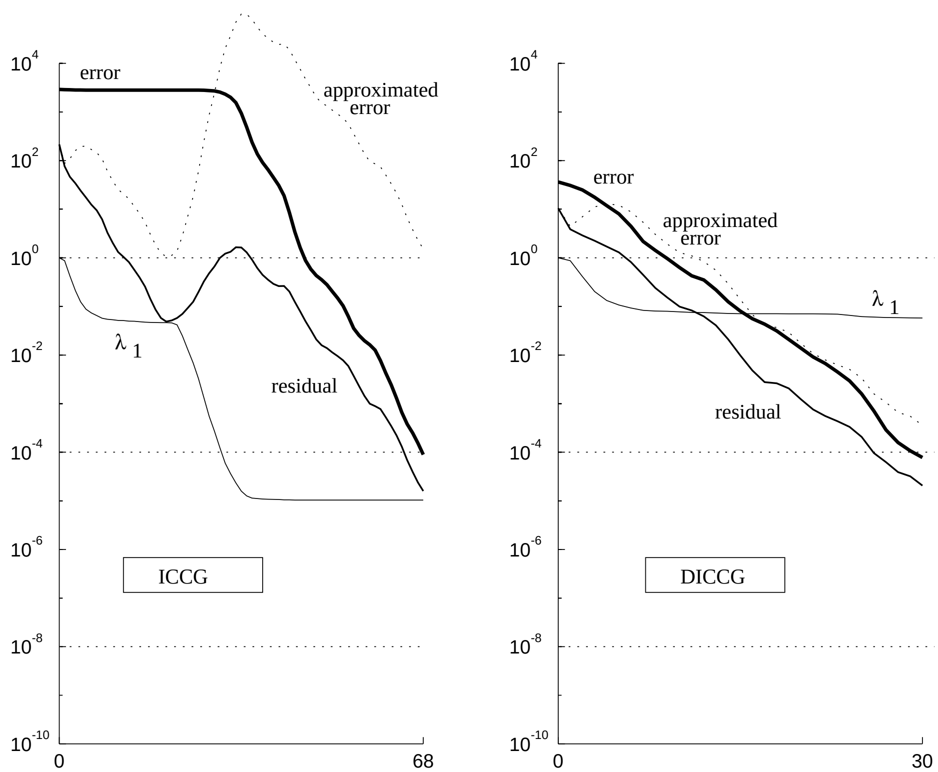
$$-\nabla \cdot (A \nabla u) = F, \tag{2}$$

where the coefficients and geometry of the problem are:



A groundwater flow problem (2)

The low permeable layer ($A = 10^{-5}$) and the jump in permeabilities between the two sand sections lead to a 'small' eigenvalue. Therefore we apply DICCG where the projection vector is 1 in the central section and 0 on the outer boundary and satisfies (2) in the other sections. The convergence of ICCG and DICCG is given below.



Conclusions

DICCG

The presented construction of the projection vectors lead to a very efficient DICCG method for the solution of complicated three-dimensional problems with large jumps in the coefficients.

Stability

DICCG is stable with respect to perturbations of the non-constant parts of the projection vectors.

Reference

C. Vuik, A. Segal, and J.A. Meijerink
An efficient preconditioned CG method for the solution of layered problems with extreme contrasts in the coefficients
J. Comp. Physics, 1999, to appear