Deflation type methods combined with shifted Laplace preconditioners for the Helmholtz equation

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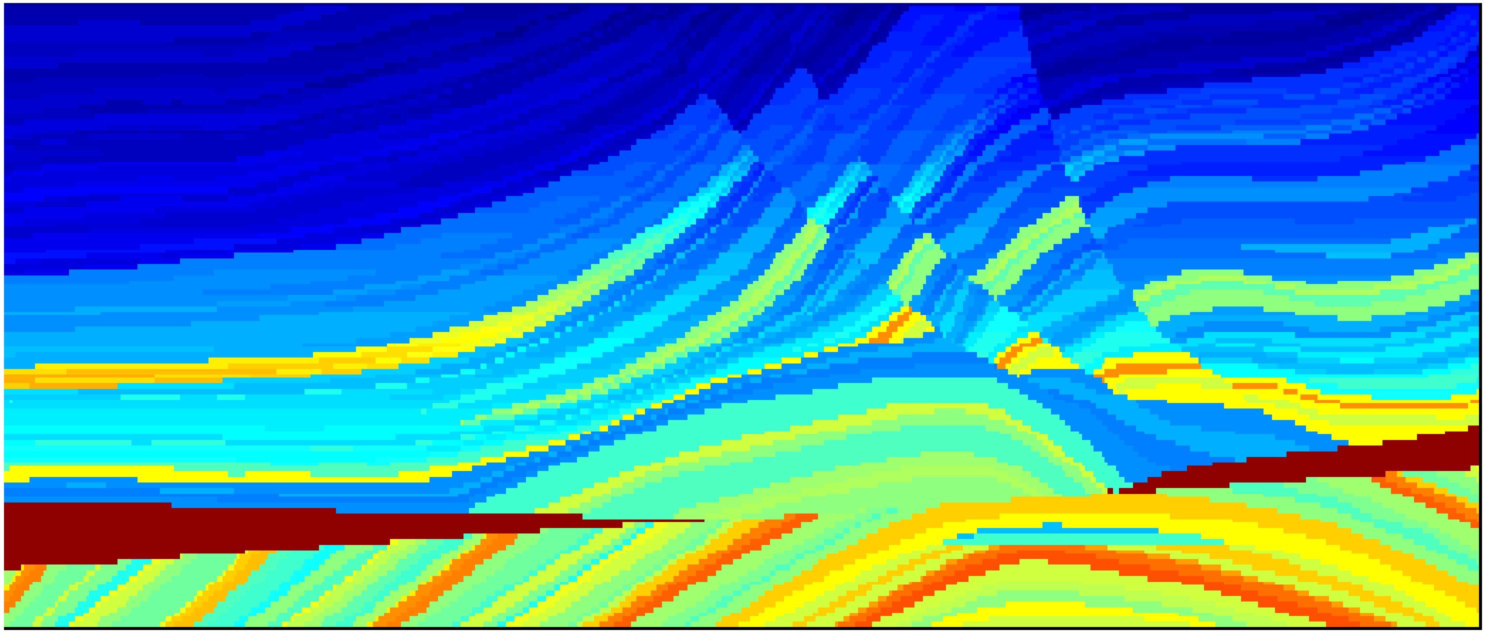
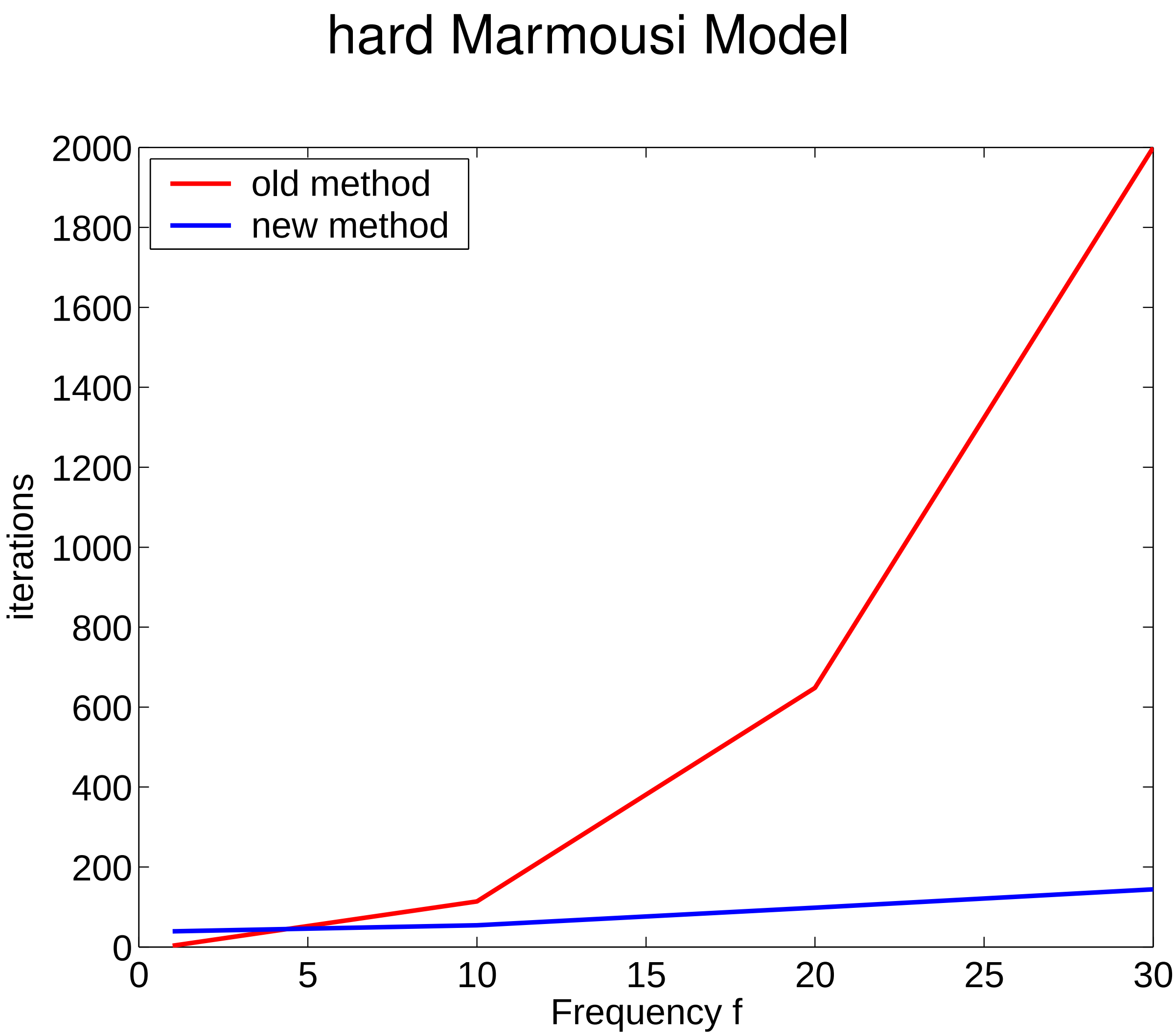
Application: geophysical survey

hard Marmousi Model



Application: geophysical survey



Contents

1. Problem: The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta\mathbf{u}(x,y) - k^2(x,y)\mathbf{u}(x,y) = \mathbf{g}(x,y) \ \text{ in } \Omega$$

$\mathbf{u}(x,y)$ is the pressure field,
$\mathbf{k}(x,y)$ is the wave number,
$\mathbf{g}(x,y)$ is the point source function and
Ω is the domain. Absorbing boundary conditions are used on Γ.

$$\frac{\partial\mathbf{u}}{\partial n} - \iota\mathbf{u} = 0$$

n is the unit normal vector pointing outwards on the boundary.

Perfectly Matched Layer (**PML**) and Absorbing Boundary Layer (**ABL**)

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Problem description

- Second order Finite Difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4 - k^2h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $A_h u_h = g_h$: properties
  Sparse & complex valued
  Symmetric & Indefinite for large k
- For high resolution a very fine grid is required: $30 - 60$ gridpoints per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A_h$ is extremely large!
- Traditionally solved by a Krylov subspace method, which exploits the sparsity.

2. Shifted Laplace Preconditioner

Laplace operator	Bayliss and Turkel, 1983
Definite Helmholtz	Laird, 2000
Shifted Laplace	Y.A. Erlangga, C. Vuik and C.W.Oosterlee, 2003

Shifted Laplace preconditioner

$$M \equiv -\Delta - (\beta_1 - \mathrm{i}\beta_2)k^2, \;\; \beta_1, \beta_2 \in \mathbb{R}.$$

If $\beta_1 \leq 0$ holds than M is a (semi) definite operator.

$\rightarrow \beta_1, \beta_2 = 0$	:	Bayliss and Turkel
$\rightarrow \beta_1 = 1, \beta_2 = 0$	:	Laird
$\rightarrow \beta_1 = -1, \beta_2 = 0.5$	:	Y.A. Erlangga, C. Vuik and C.W. Oosterlee

Numerical results for a wedge problem

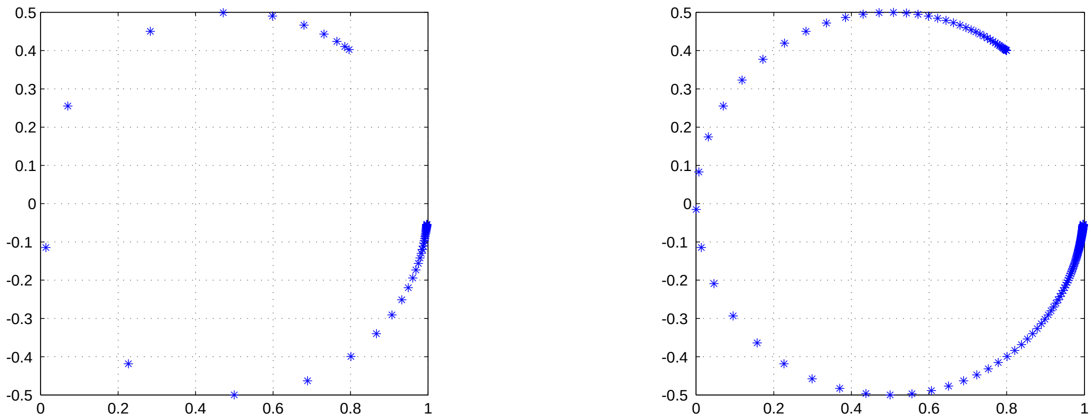
k_2	10	20	40	50	100
grid	32^2	64^2	128^2	192^2	384^2
No-Prec	201(0.56)	1028(12)	5170(316)	–	–
ILU(A,0)	55(0.36)	348(9)	1484(131)	2344(498)	–
ILU(A,1)	26(0.14)	126(4)	577(62)	894(207)	–
ILU(M,0)	57(0.29)	213(8)	1289(122)	2072(451)	–
ILU(M,1)	28(0.28)	116(4)	443(48)	763(191)	2021(1875)
MG(V(1,1))	13(0.21)	38(3)	94(28)	115(82)	252(850)

Shifted Laplace Preconditioner Spectrum

- Eigenvalues of the preconditioned operator are bounded by 1
- Small eigenvalues move to zero, as k increases.

Spectrum of $M^{-1}A$, where $(\beta_1, \beta_2) = (1, 0.5)$

$k = 30$ $k = 120$



3. Second Level Preconditioners

Number of GMRES iterations. Shifts in the preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 64$	4/10	**6/17**	8/28	12/36	18/45	173/163
$n = 96$	3/10	5/17	**7/27**	9/35	12/43	36/97
$n = 128$	3/10	4/17	6/27	**7/35**	9/43	36/85
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Erlangga and Nabben, 2008, seems to be independent of k.

with / without deflation.

Deflation: or two-grid method

Deflation, a projection preconditioner

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \;\; \text{and} \;\; E = Z^T AZ$$

where,

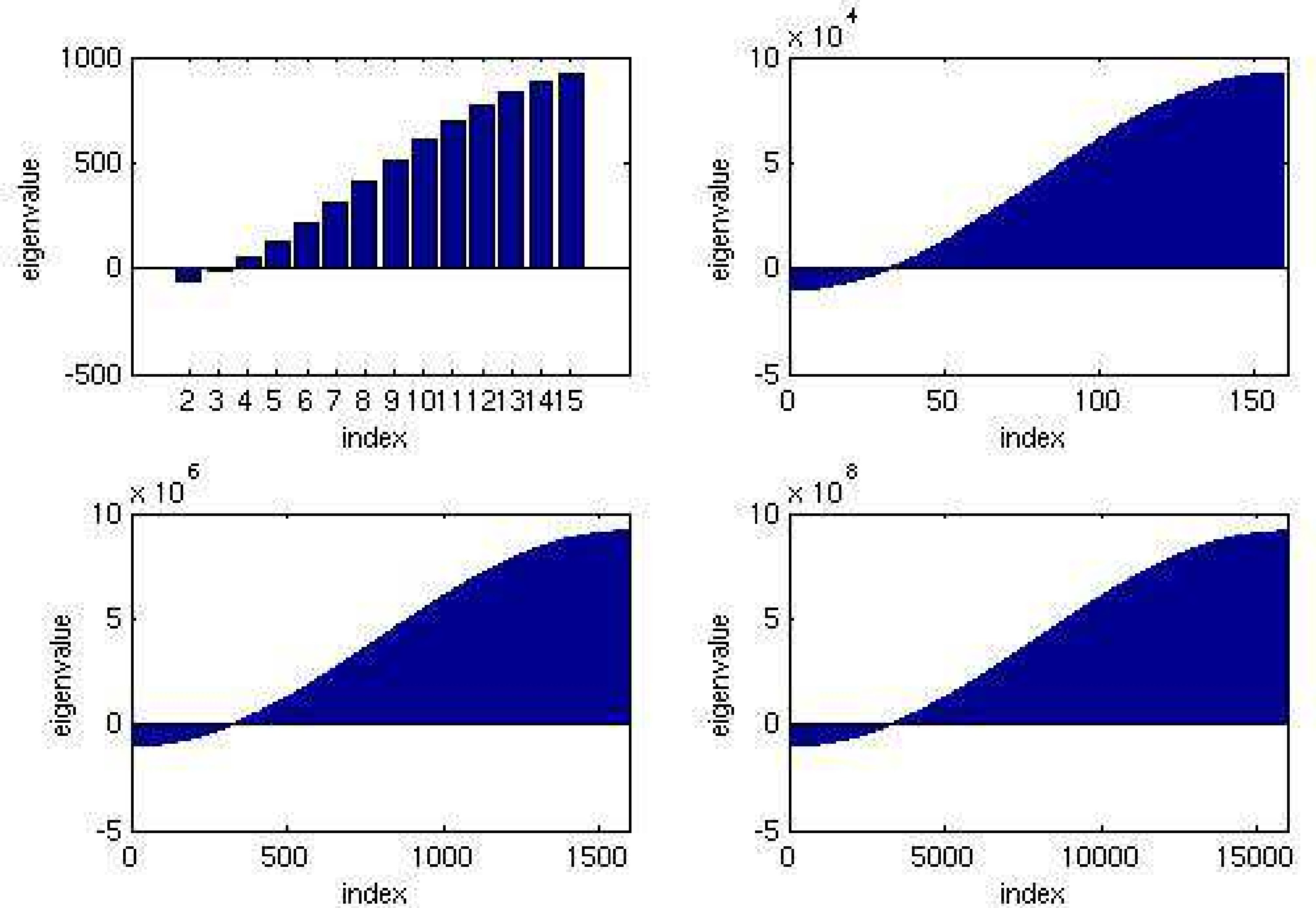
$$Z \in R^{n\times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rank(Z) = r \leq n$$

Along with a traditional preconditioner M, deflated preconditioned system reads

$$PM^{-1}Au = PM^{-1}g.$$

Deflation vectors shifted the eigenvalues to zero.

Spectrum as function of k



Deflation for Helmholtz

With choice of multigrid inter-grid transfer operator (Prolongation) as deflation matrix, i.e. $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

$$P_h = I_h - A_h Q_h, \quad \text{with} \quad Q_h = I_h^{2h} A_{2h}^{-1} I_{2h}^h \ \text{ and } A_{2h} = I_{2h}^h A_h I_h^{2h}$$

where

P_h can be interpreted as a coarse grid correction and

Q_h as the coarse grid operator

Deflation: ADEF1

Deflation can be implemented combined with SLP M_h,

$$M_h^{-1}P_hA_hu_h = M_h^{-1}P_hg_h$$

$A_hu_h = g_h$ is preconditioned by two-level preconditoner $M_h^{-1}P_h$.

For large problems, A_{2h} is too large to invert exactly.
Inversion of A_{2h} is sensitive, since P_h deflates the spectrum to zero.

To do is: Solve A_{2h} iteratively to required accuracy on certain levels, and shift the deflated spectrum to λ_h^{max} by adding a shift in two level preconditioner. This leads to the **ADEF1** preconditioner

$$P_{(h,ADEF1)} = M_h^{-1}P_h + \lambda_h^{max}Q_h$$

Deflation: MLKM

Multi Level Krylov Method [a], take $\hat{A}_h = M_h^{-1}A_h$, and define $\hat{P}_h$ by using $\hat{A}_h$ (instead of A_h) will be

$$\hat{P}_h = I_h - \hat{A}_h\hat{Q}_h,$$

where

$$\hat{Q}_h = I_h^{2h}\hat{A}_{2h}^{-1}I_{2h}^h \text{ and } \hat{A}_{2h} = I_{2h}^h\hat{A}_hI_h^{2h} = I_{2h}^h(M_h^{-1}A_h)I_h^{2h}$$

Construction of coarse matrix A_{2h} at level $2h$ costs inversion of preconditioner at level h.

Approximate A_{2h}

Ideal	**Practical**
$A_{2h} = I_{2h}^h(M_h^{-1}A_h)I_h^{2h}$	$A_{2h} = I_{2h}^h(M_h^{-1}A_h)I_h^{2h}$
	$A_{2h} \approx I_{2h}^hI_h^{2h}M_{2h}^{-1}A_{2h}$

[a] Erlangga, Y.A and Nabben R., ETNA 2008

4. Fourier Analysis

Dirichlet boundary conditions for analysis.
With above deflation,

$$\textbf{spec}(PM^{-1}A) = f(\beta_1, \beta_2, k, h)$$

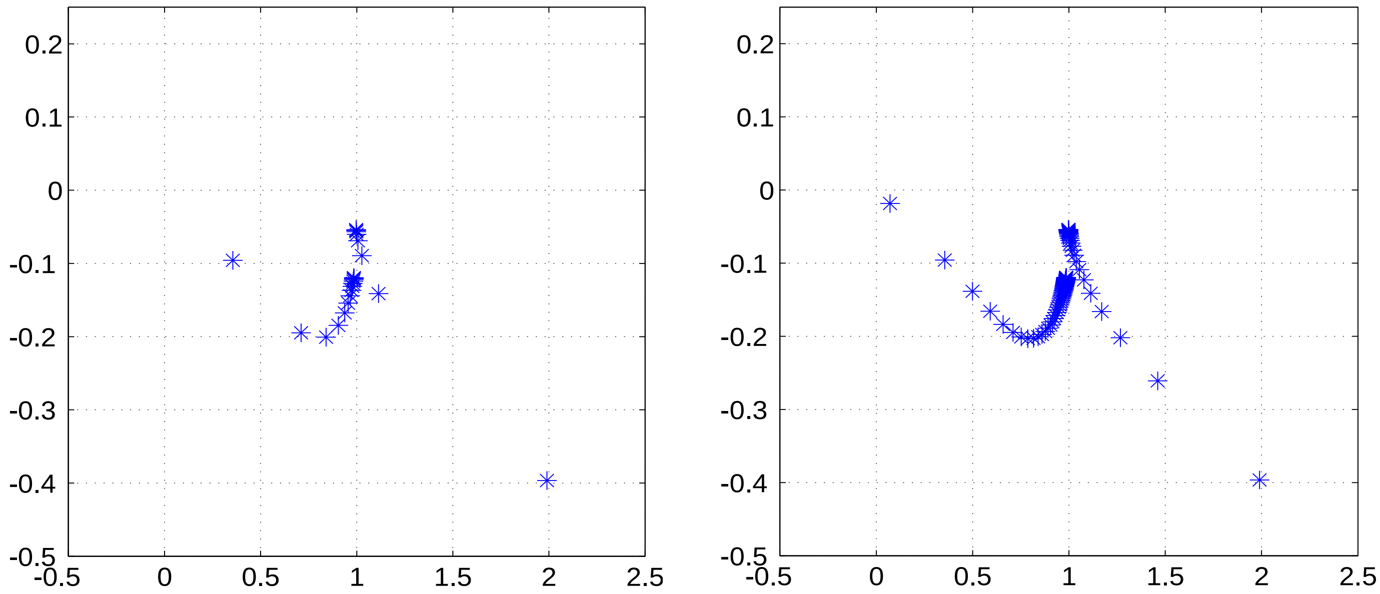
is a complex valued function.
Setting $kh = 0.625$,

- Spectrum of $PM^{-1}A$ with shifts $(1, 0.5)$ is clustered around 1 with a few outliers.
- Spectrum remains almost the same, when the imaginary shift for the preconditioner is varied from 0.5 to 1.

Fourier Analysis

ADEF1: Analysis shows spectrum clustered around 1 with few outliers.

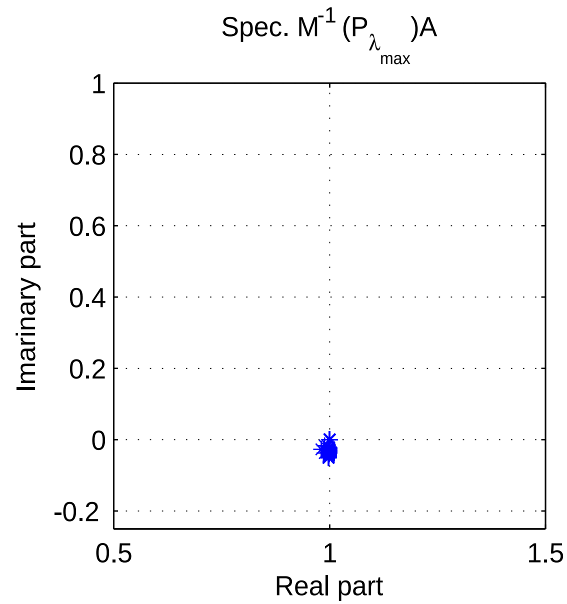
$k = 30$ $\qquad$ $k = 120$



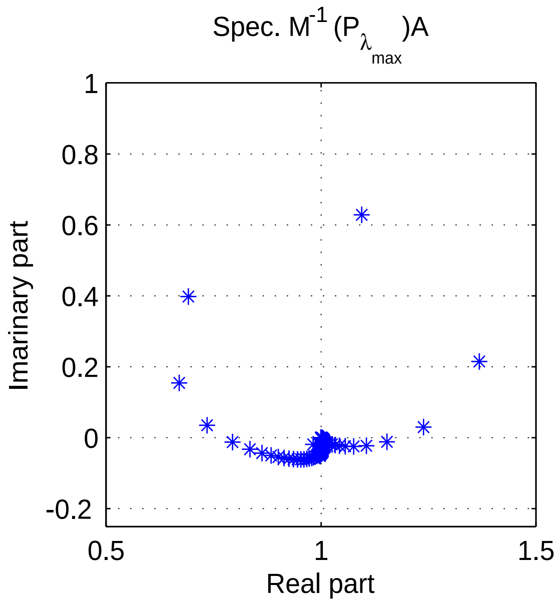
Fourier Analysis

Spectrum of Helmholtz preconditioned by MLKM [b],
$k = 160$ and 20 gp/wl

Ideal

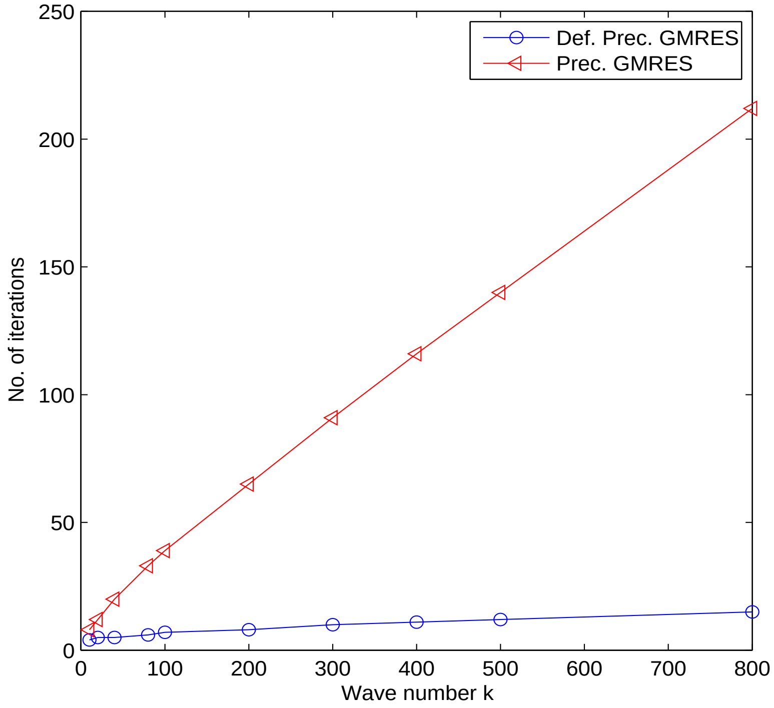


Practical



[b] Two-level

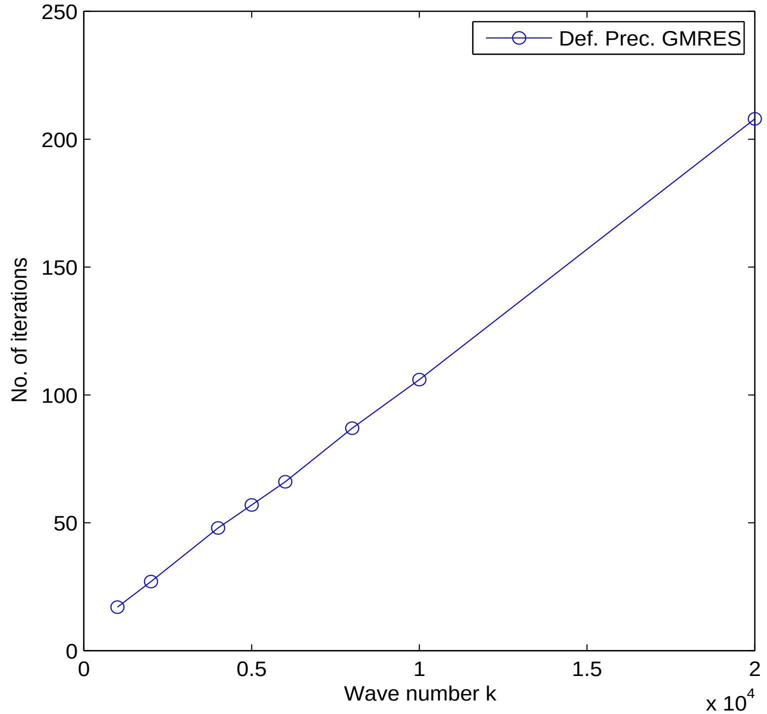
5. Numerical results



- Number of GMRES iterations for the 1D Helmholtz equation

$$10 \leq k \leq 800$$

Numerical results



Number of GMRES iterations for the 1D Helmholtz equation

$$1000 \leq k \leq 20000$$

Numerical results

Number of GMRES outer-iterations in multilevel algorithm.
$(\beta_1, \beta_2) = (1, 0.5)$ $kh = .3125$ or 20 gp/wl
and SLP approximated by multigrid Vcyle V(1,1)

Grid	$k = 10$	$k = 20$	$k = 40$	$k = 80$	$k = 160$
ADEF1-V(4,2,1)	9	11	16	27	100+
ADEF1-V(6,2,1)	9	10	14	21	47
ADEF1-V(8,2,1)	9	10	13	20	38
ADEF1-V(8,3,2)	9	10	13	19	37

ADEF1-V(8,2,1), a multilevel solver where 8 and 2 Krylv iterations performed on 2^{nd} and

3^{rd} levels and 1 iteration on further levels and Vcycle approximates SLP.

Results

Petsc solve-time in Seconds; a Two-level solver.

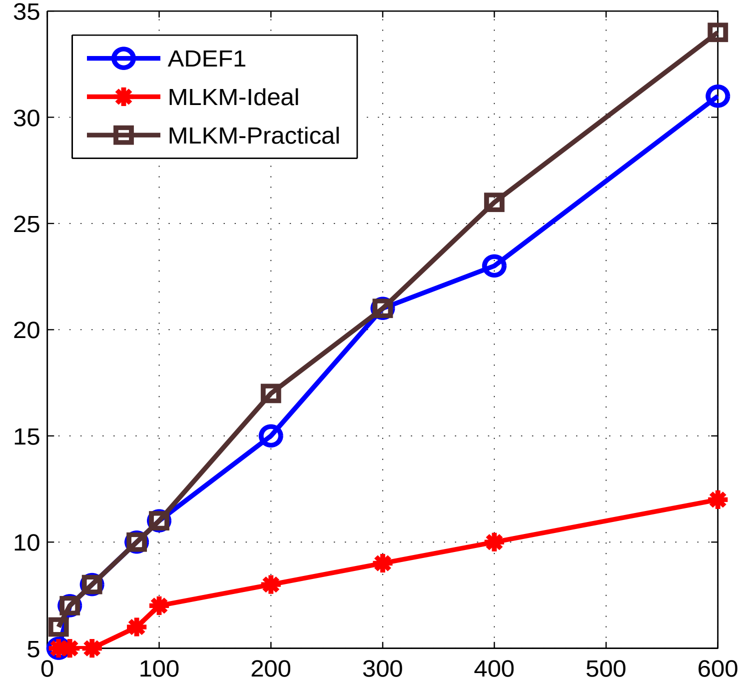
Solver	20	40	80	120	160	200
SLP	0.01(23)	0.24(54)	2.62(113)	11.60(168)	33.59(222)	83.67(274)
ADEF1/SLP	0.03(10)	0.14(14)	0.82(23)	2.92(37)	8.98(61)	23.13(87)

SLP : GCR preconditioned with SLP $M(1,1)$.

Def/SLP: Deflated and preconditioned GCR.

Grid resolution is such that there are 10 grid points per wavelength.

Numerical results



Comparison of number of iterations by ADEF1 and MLKM.

Numerical results

3D Helmholtz on unit cube with Sommerfeld b.c. on all faces.

	Wavenumber k							
Solver Type	5	10	15	20	30	40	60	80
SL Prec.	11	15	21	29	47	74	118	185
ADEF1-F(8,2,1)	9	10	11	11	13	16	22	28
ADEF1-V(8,2,1)	11	15	21	28	44	67	101	153

SL Prec. : Only shifted Laplace preconditioner

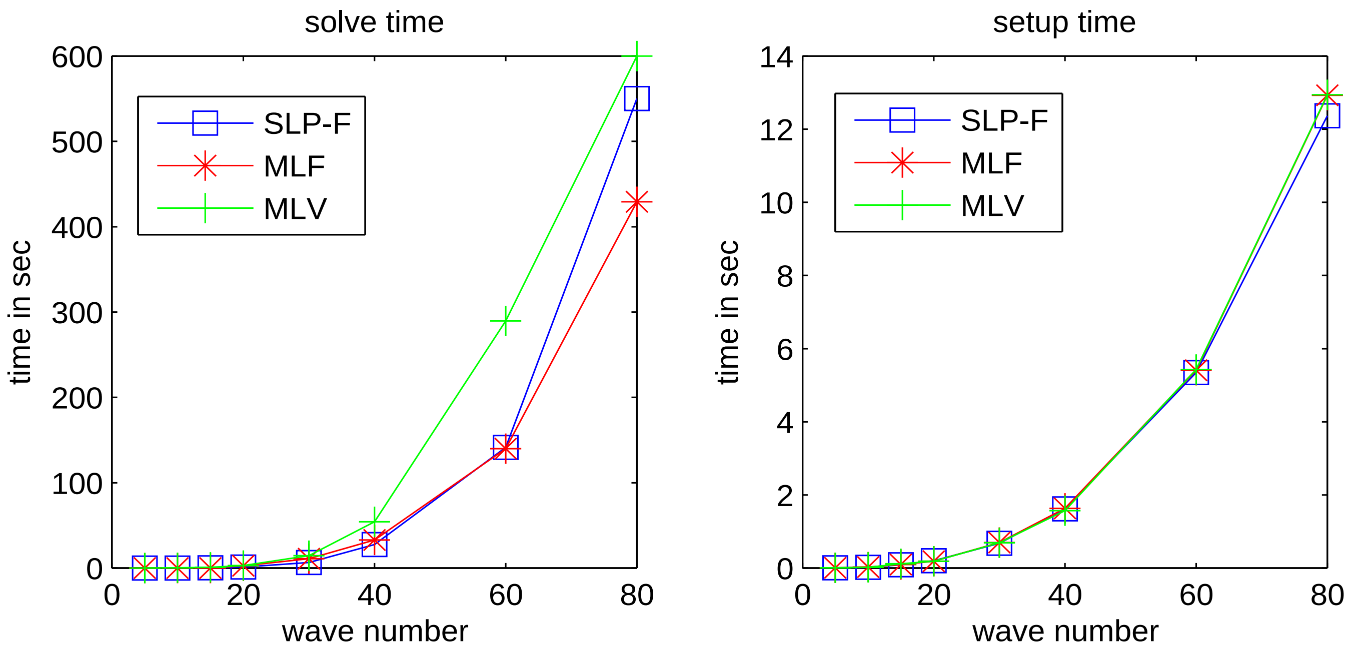
ADEF1-F : Multilevel solver , Fcycle for slp.

ADEF1-V : Multilevel solver , Vcycle for slp.

Numerical results

Multilevel solver is coded in Petsc.

3D Helmholtz on unit cube with constant wavenumber.



ADEF1 solve time and Setup time.

Conclusions

- Without deflation, when imaginary shift is increased in SLP, spectrum remains bounded above 1, but lower part moves to zero.
- With deflation the convergence is nearly independent of the imaginary shift.
- With deflation the convergence is initially weakly depending on k. For large k is scales again linearly.
- With deflation the CPU time is less than without deflation.
- The convergence of ADEF1 and the practical variant of MLKM are similar.

References

- Y.A. Erlangga and C.W. Oosterlee and C. Vuik A Novel Multigrid Based Preconditioner For Heterogeneous Helmholtz Problems SIAM J. Sci. Comput.,27, pp. 1471-1492, 2006
- M.B. van Gijzen, Y.A. Erlangga and C. Vuik. Spectral analysis of the discrete Helmholtz operator preconditioned with a shifted Laplacian. SIAM J.of Sc. Comp. 2007.
- Y.A. Erlangga and R. Nabben. On a multilevel Krylov method for the Helmholtz equation preconditioned by shifted Laplacian. ETNA, 2008.
- H. Knibbe and C.W. Oosterlee and C. Vuik GPU implementation of a Helmholtz Krylov solver preconditioned by a shifted Laplace multi-grid method. Journal of Computational and Applied Mathematics, 236, pp. 281-293, 2011
- A.H. Sheikh, D. Lahaye and C. Vuik. On the convergence of shifted Laplace preconditioner combined with multi-grid deflation. NLAA 2013 http://onlinelibrary.wiley.com/doi/10.1002/nla.1882/abstract
- http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_helmholtz.html