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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday August 15 2013, 18:30-21:30

1. We consider the following method for the integration of the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$

$$\begin{cases} w^*_{n+1} = w_n + hf(t_n, w_n) \\ \\ w_{n+1} = w_n + h\left(a_1 f(t_n, w_n) + a_2 f(t_{n+1}, w^*_{n+1})\right) \end{cases} \tag{1}$$

a Show that the local truncation error of the above method has order $O(h)$ if $a_1 + a_2 = 1$. Which value for a_1 and a_2 will give a local truncation error of order $O(h^2)$? (3 pt.)

b Demonstrate that for general values of a_1 and a_2 the amplification factor is given by

$$Q(h\lambda) = 1 + (a_1 + a_2)h\lambda + a_2(h\lambda)^2. \tag{2}$$

(2 pt.)

c Consider a real valued $\lambda < 0$ and $(a_1 + a_2)^2 - 8a_2 < 0$. Derive the condition for stability, to be fullfilled by h. (2 pt.)

d Consider the following system

$$\begin{cases} y_1' = -y_1 y_2, \\ y_2' = y_1 y_2 - y_2, \end{cases} \tag{3}$$

Show that the Jacobian of the right hand side of the above system (which is used for the linearization of the above system) for the initial condition $y_1(0) = 1$ and $y_2(0) = 2$ is given by

$$\begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}.$$

(1.5 pt.)

e We apply the numerical method in equation (1) for the case that $a_1 = a_2 = 1/2$ to system (3). Is the method stable near the initial condition $y_1(0) = 1$ and $y_2(0) = 2$, and step size $h = 1$ (+ motivation)? (1.5 pt.)

[0]**please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. We consider the convection–diffusion equation with Dirichlet boundary conditions:

$$(P_1)\begin{cases} -u'' + u' = 0, & 0 < x < 1, \\ u(0) = 0, & u(1) = 1, \end{cases} \tag{4}$$

where $u = u(x)$ and $u' = \frac{du}{dx}$.

a Show that
$$u(x) = \frac{e^x - 1}{e - 1}, \tag{5}$$
is the exact solution to boundary value problem (P_1). (1.5 pt.)

b We solve boundary value problem (P_1) using finite differences, upon setting $x_j = jh$, $(n+1)h = 1$, where h denotes the uniform stepsize. Give a discretization method (+proof) where the truncation error is of order $O(h^2)$. Take the boundary conditions into account. (3 pt.)

c Give a (physical or mathematical) motivation why non–monotonic (oscillatory) numerical solutions to (P_1) should be considered unreliable. (1.5pt.)

d We numerically integrate the function f of which we only have the values listed in Table 1 at some discrete points.

Table 1: Values of the function $f(x)$ at discrete points.

x	$f(x)$
0	0
0.1	0.01
0.2	0.04
0.3	0.09

The Trapezoidal Rule over an interval $[a, b]$ is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right). \tag{6}$$

i Let the derivatives of f up to at least second order be continuous. Derive the local truncation error for the Trapezoidal integration rule. *Hint: You can use truncation error for linear interpolation.* (2 pt.)

ii Derive the repeated Trapezoidal Rule and apply this rule to estimate $\int_0^{0.3} f(x)dx$. (2 pt.)