DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU and CTB2400)
Thursday August 14 2014, 18:30-21:30

1. In this assignment predictor-corrector methods of the following type are considered:

$$\begin{aligned} k_1 &= hf(t_n, w_n) \\ k_2 &= hf(t_n + h, w_n + k_1) \\ w_{n+1} &= w_n + \beta\ k_1 + (1-\beta)\ k_2. \end{aligned} \tag{1}$$

a) Show, for the general equation $y' = f(t, y)$, that the local truncation error is $O(h^2)$ for $\beta = \frac{1}{2}$ and $O(h)$ for other values of β. (3 pt)

b) Determine the amplification factor for arbitrary β. (2 pt)

c) Method (1) can be applied to the system

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}. \tag{2}$$

Show that the stepsize h must satisfy

$$h^2 < \frac{1-2\beta}{(1-\beta)^2} \tag{3}$$

for stable integration of (2). For which values of β is stable integration possible? (2 pt)

d) Show that the stability bound (3) becomes optimal for $\beta = 0$ and give this stability bound. (Hint: the derivative of $\frac{1-2\beta}{(1-\beta)^2}$ is equal to $\frac{-2\beta}{(1-\beta)^3}$.) (1 pt)

e) Let $\beta = 0$, and let the initial values be given by $y_1(0) = 1$ and $y_2(0) = 1$. Use $h = 0.5$ to compute the numerical solution at the next time step.

[0] **please turn over, For the answers of this test we refer to:**
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html

2. We analyze Lagrangian interpolation. For given points x_0, x_1 …, x_n, with their respective function values $f(x_0)$, $f(x_1)$, …, $f(x_n)$, the interpolatory polynomial $p_n(x)$ is given by

$$\begin{aligned} p_n(x) &= \sum_{i=0}^{n} f(x_i)L_i(x), \text{ with} \\ L_i(x) &= \frac{(x-x_0)(\ldots)(x-x_{i-1})(x-x_{i+1})(\ldots)(x-x_n)}{(x_i-x_0)(\ldots)(x_i-x_{i-1})(x_i-x_{i+1})(\ldots)(x_i-x_n)}. \end{aligned} \tag{4}$$

Further, the following measured values have been given in tabular form:

i	x_i	$f(x_i)$
0	0	1
1	1	2
2	2	4

(a) Give the linear Lagrangian interpolatory polynomial with nodes x_0 and x_1. (1pt.)

(b) Give the quadratic Lagrangian interpolatory polynomial with nodes x_0, x_1 and x_2. (2pt.)

(c) Approximate $f(0.5)$ both by using linear and quadratic Lagrangian interpolation. (2pt.)

(d) Suppose that the function values in the table contain measurement errors with a magnitude of at most ε.Show that the error, as a consequence of the inaccuracy of the measured data, is bounded as long as we interpolate between the nodes x_0 and x_1 if we use linear interpolation. (1pt.)

(e) The following iteration process is given $x_{n+1} = g(x_n)$, with

$$g(x_n) = x_n + h(x_n)(x_n^3 - 3),$$

where h is a continuous function with $h(x) \neq 0$ for each $x \neq 0$. If this process converges, to which limit p does it converge? (1pt.)

(f) Consider three possible choices for $h(x)$:

i. $h_1(x) = -\frac{1}{x^4}$

ii. $h_2(x) = -\frac{1}{x^2}$

iii. $h_3(x) = -\frac{1}{3x^2}$

For which choice does the process not converge? For which choice is the convergence the fastest? Motivate your answer. (2pt.)

(g) p is the root of a given function f. $\hat{f}$ is the function perturbed by measurement errors. It is given that $|\hat{f}(x) - f(x)| \leq \epsilon_{max}$ for all x. Show that the root $\hat{p}$ from $\hat{f}$ satisfies the following inequality $|\hat{p} - p| \leq \frac{\epsilon_{max}}{|f'(p)|}$. (1pt.)