DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU AESB2210)
Thursday April 16 2015, 18:30-21:30

1. To integrate the initial value problem $y' = f(t, y)$, with $y(t^0) = y^0, a$ we consider the Trapezoidal Rule

$$w^{n+1} = w^n + \frac{h}{2}(f(t^n, w^n) + f(t^{n+1}, w^{n+1})), \tag{1}$$

and the Modified Euler Method.

[a] Compute the amplification factors of both methods. (2pt.)

[b] Show that the local truncation error of both methods is of order $O(h^2)$.

Hint: It is allowed to use the test equation for both the Trapezoidal Rule and Modified Euler Method. Further, note that $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + O(x^4)$ *and if* $|x| < 1$ *then* $\frac{1}{1-x} = 1 + x + x^2 + x^3 + O(x^4)$. (3pt.)

We apply both methods to the initial value problem

$$y'' = -4y + 2t, \quad y(0) = 1, \quad y'(0) = 0. \tag{2}$$

[c] Show that this initial value problem can be rewritten as the following system of first–order equations

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2t \end{pmatrix}, \tag{3}$$

with initial condition $y_1(0) = 1$ and $y_2(0) = 0$. (1pt.)

[d] Use $h = \frac{1}{2}$ to compute w^1 (one time–step) using both the Trapezoidal Rule and the Modified Euler Method. (2pt.)

[e] Which of the two methods do you prefer to apply to the present initial value problem (in assignment [c–d])? Motivate your choice in terms of accuracy, stability and workload. (2pt.)

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[0]**please turn over, For the answers of this test we refer to:**
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html

2. We consider the following boundary value problem (second-order differential equation with boundary conditions at $x = 0$ and $x = 1$):

$$\begin{cases} -y'' + xy - x^3 + 2 = 0, \\ y'(0) = 0, \\ y(1) = 1. \end{cases} \tag{4}$$

   (a) Show that $y(x) = x^2$ is the solution of problem (4). (1pt.)

   (b) Let h be the step size. Give a discretization with a local truncation error of $O(h^2)$ (+ proof). Use a virtual grid node near $x = 0$. (2pt.)

   (c) Use a step size of $h = 1/3$ to derive the system of equations $Ay_h = b$. Take care of the boundary conditions. The system must have three unknowns and three equations, i.e. A is a 3×3 matrix and y_h and b are 1×3 column vectors. (2pt.)

   (d) Give the numerical solution y_h for the step size $h = 1/3$. Why is the error $e(x) = y_h(x) - y(x)$ zero at all grid points $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$ and $x_3 = 1$? (1pt.)

Next, we are interested in $\int_0^1 y(x)dx$, which will be approximated numerically.

   (e) Give the Rectangle Rule I^R, the corresponding composed integration rule $I^R(h)$ and compute the approximate integral $\int_0^1 y(x)dx$ with $h = 1/3$. (1pt.)

   (f) Repeat part (e) for the Trapezoidal Rule (I^T and $I^T(h)$). (1pt.)

   (g) If one approximates $\int_0^1 y(x)dx$, the magnitude of the error of the composed integration rules (ε_R and ε_T for the Rectangle and Trapezoidal Rule, respectively) is bounded by

$$\varepsilon_R \leq \frac{h}{2} \max_{x \in [0,1]} |y'(x)|, \quad \varepsilon_T \leq \frac{h^2}{12} \max_{x \in [0,1]} |y''(x)|. \tag{5}$$

   Which method do you recommend if the number of integration points is large? Give a proper motivation. (2pt.)