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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU AESB2210 CTB2400)
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1. (a) The local truncation error is given by

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t} \tag{1}$$

where z_{n+1} is the result of applying the method once with starting solution y_n. Here we obtain y_{n+1} by a Taylor expansion around t_n:

$$y_{n+1} = y_n + \Delta t y'(t_n) + \frac{(\Delta t)^2}{2} y''(t_n) + O((\Delta t)^3). \tag{2}$$

For z_{n+1}, we obtain, after substitution of the predictor step for z^*_{n+1} into the corrector step

$$z_{n+1} = y_n + \Delta t \left((1-\mu) f(t_n, y_n) + \mu f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))\right) \tag{3}$$

After a Taylor expansion of $f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))$ around (t_n, y_n) one obtains:

$$z_{n+1} = y_n + \Delta t \left((1-\mu) f(t_n, y_n) + \mu (f(t_n, y_n) + \Delta t (\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n) \frac{\partial f(t_n, y_n)}{\partial y})) + O((\Delta t)^2) \right). \tag{4}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{5}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} y'(t_n) \tag{6}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n) \tag{7}$$

This implies that $z_{n+1} = y_n + \Delta t y'(t_n) + \mu (\Delta t)^2 y''(t_n) + O((\Delta t)^3)$. Subsequently, it follows that

$$y_{n+1} - z_{n+1} = O((\Delta t)^2), \text{ and, hence } \tau_{n+1}(\Delta t) = \frac{O((\Delta t)^2)}{\Delta t} = O(\Delta t) \text{ for } 0 \le \mu \le 1, \tag{8}$$

$$y_{n+1} - z_{n+1} = O((\Delta t)^3), \text{ and, hence } \tau_{n+1}(\Delta t) = \frac{O((\Delta t)^3)}{\Delta t} = O((\Delta t)^2) \text{ for } \mu = \frac{1}{2}. \tag{9}$$

(b) Consider the test equation $y' = \lambda y$, then, herewith, one obtains

$$\begin{aligned}
&w^*_{n+1} = w_n + \lambda\Delta t w_n = (1+\lambda\Delta t)w_n,\\
&w_{n+1} = w_n + ((1-\mu)\lambda\Delta t w_n + \mu\lambda\Delta t w^*_{n+1}) =\\
&= w_n + ((1-\mu)\lambda\Delta t w_n + \mu\lambda\Delta t(w_n + \lambda\Delta t w_n)) = (1+\lambda\Delta t + \mu(\lambda\Delta t)^2)w_n.
\end{aligned} \tag{10}$$

Hence the amplification factor is given by

$$Q(\lambda\Delta t) = 1 + \lambda\Delta t + \mu(\lambda\Delta t)^2. \tag{11}$$

(c) Doing one step with the given method with $\Delta t = \frac{1}{2}$ and $\mu = \frac{1}{2}$ leads to the following steps:
Predictor:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -0 + \cos(0) + 2 + 0 \\ 0 - 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$$

Corrector:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\left(\frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\frac{3}{2} + \cos(\frac{3}{2}) + 2\cdot\frac{1}{2} + \frac{1}{2} \\ \frac{3}{2} - (\frac{1}{2})^2 \end{pmatrix}\right)$$

which can be written as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 + \frac{3}{4} - \frac{3}{8} + \frac{1}{4}\cos(\frac{3}{2}) + \frac{3}{8} \\ 1 - \frac{1}{4} + \frac{3}{8} - \frac{1}{16} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}\cos(\frac{3}{2}) \\ \frac{17}{16} \end{pmatrix} = \begin{pmatrix} 0.7677 \\ 1.0625 \end{pmatrix}$$

(d) In order to compute the Jacobian, we note that the right-hand side of the non linear system can be noted by:

$$f_1(x_1, x_2) = -x_1 + \cos x_1 + 2x_2 + t$$

$$f_2(x_1, x_2) = x_1 - x_2^2$$

From the definition of the Jacobian it follows that:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -1 - \sin x_1 & 2 \\ 1 & -2x_2 \end{pmatrix}.$$

Substitution of $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ shows that

$$J = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

(e) For the stability it is sufficient to check that $|Q(\lambda_i \Delta t)| \leq 1$ for all the eigenvalues of the Jacobian matrix. It is easy to see that the eigenvalues of the Jacobian matrix are $\lambda_1 = -3$ and $\lambda_2 = 0$.

For the choice $\mu = 0$ we note that the method is equal to the Euler Forward method. For real eigenvalues the Euler Forward method is stable if $\Delta t \leq \frac{-2}{\lambda}$. Since $\lambda_1 = -3$ and $\lambda_2 = 0$ we know that the method is stable if $\Delta t \leq \frac{-2}{-3} = \frac{2}{3}$ (another option is to derive the values of Δt such that $|Q(\lambda_i \Delta t)| \leq 1$ by using the description of $Q(\lambda \Delta t)$)

For the choice $\mu = \frac{1}{2}$ we use the expression

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2$$

For $\lambda_2 = 0$ it appears that $Q(\lambda_2 \Delta t) = 1$ so the inequality is satisfied for all Δt. For $\lambda_1 = -3$ we have to check the following inequalities:

$$-1 \leq 1 - 3\Delta t + \frac{9}{2}(\Delta t)^2 \leq 1$$

For the left-hand inequality we arrive at

$$0 \leq \frac{9}{2}(\Delta t)^2 - 3\Delta t + 2$$

It appears that the discriminant $9 - 4 \cdot \frac{9}{2} \cdot 2$ is negative, so there are no real roots which implies that the inequality is satisfied for all Δt.

For the right-hand inequality we get

$$-3\Delta t + \frac{9}{2}(\Delta t)^2 \leq 0$$

$$\frac{9}{2}(\Delta t)^2 \leq 3\Delta t$$

so

$$\Delta t \leq \frac{2}{3}$$

(another option is to see that for $\mu = \frac{1}{2}$ the method is equal to the modified Euler method, and remember that this method is stable for real eigenvalues if $\Delta t \leq \frac{-2}{\lambda}$)

2. (a) First, we check the boundary conditions:

$$u(0) = 0 - \frac{1 - e^0}{1 - e} = \frac{1 - 1}{1 - e} = 0, \quad u(1) = 1 - \frac{1 - e^1}{1 - e} = 0. \qquad (12)$$

Further, we have

$$u'(x) = 1 + \frac{e^x}{1-e}, \tag{13}$$

$$u''(x) = \frac{e^x}{1-e}. \tag{14}$$

Hence, we immediately see

$$-u''(x) + u'(x) = -\frac{e^x}{1-e} + 1 + \frac{e^x}{1-e} = 1. \tag{15}$$

Hence, the solution $u(x) = 1 - \frac{1-e^x}{1-e}$ satisfies the differential and the boundary conditions, and therewith $u(x)$ is the solution to the boundary value problem (uniqueness can be demonstrated in a straightforward way, but this was not asked for).

(b) The domain of computation, being $(0,1)$, is divided into subintervals with mesh points, we set $x_j = j\Delta x$, where we use n unknowns, such that $x_{n+1} = (n+1)\Delta x = 1$. We are looking for a discretization with an error of second order, $O((\Delta x)^2)$. To this extent, we use the following central differences approximation at x_j:

$$u'(x_j) \approx \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x}, \text{ for } j \in \{1, \ldots, n\}. \tag{16}$$

We note that the above formula can be derived formally by writing the derivative as

$$u'(x_j) = \frac{\alpha_0 u(x_{j-1}) + \alpha_1 u(x_j) + \alpha_2 u(x_{j+1})}{\Delta x}, \tag{17}$$

and solve α_0, α_1 and α_2 from checking the zeroth, first and second order derivatives of $u(x)$. Further, the second order derivative is approximated by

$$u''(x_j) \approx \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{(\Delta x)^2}. \tag{18}$$

Since we approximate the derivatives at the point x_j, we use Taylor series expansion about x_j, to obtain:

$$\begin{aligned} u(x_{j+1}) = u(x_j + \Delta x) = u(x_j) + \Delta x u'(x_j) + \frac{(\Delta x)^2}{2} u''(x_j) + \frac{(\Delta x)^3}{6} u'''(x_j) + O((\Delta x)^4), \\ u(x_{j-1}) = u(x_j - \Delta x) = u(x_j) - \Delta x u'(x_j) + \frac{(\Delta x)^2}{2} u''(x_j) - \frac{(\Delta x)^3}{6} u'''(x_j) + O((\Delta x)^4), \end{aligned} \tag{19}$$

This gives

$$\begin{aligned} -\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{(\Delta x)^2} + \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x} = -u''(x_j) + u'(x_j) \\ + \frac{O((\Delta x)^3)}{2\Delta x} + \frac{O((\Delta x)^4)}{(\Delta x)^2} = -u''(x_j) + u'(x_j) + O((\Delta x)^2). \end{aligned} \tag{20}$$

Hence the error is second order, that is $O((\Delta x)^2)$. Next, we neglect the truncation error, and set $w_j := u(x_j)$ to get

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + \frac{w_{j+1} - w_{j-1}}{2\Delta x} = 1, \text{ for } j \in \{1, \ldots, n\}. \tag{21}$$

At the boundaries, we see for $j = 1$ and $j = n$, upon substituting $w_0 = 0$ and $w_{n+1} = 0$, respectively:

$$\begin{aligned} -\frac{w_2 - 2w_1 + 0}{(\Delta x)^2} + \frac{w_2 - 0}{2\Delta x} &= 1, \\ -\frac{0 - 2w_n + w_{n-1}}{(\Delta x)^2} + \frac{0 - w_{n-1}}{2\Delta x} &= 1. \end{aligned} \tag{22}$$

This can be rewritten more neatly as follows:

$$\begin{aligned} \frac{-w_2 + 2w_1}{(\Delta x)^2} + \frac{w_2}{2\Delta x} &= 1, \\ \frac{2w_n - w_{n-1}}{(\Delta x)^2} - \frac{w_{n-1}}{2\Delta x} &= 1. \end{aligned} \tag{23}$$

(c) The real-valued exact solution and its first and second derivative are given by

$$u(x) = x - \frac{1 - e^x}{1 - e}, \tag{24}$$

$$u'(x) = 1 + \frac{e^x}{1 - e}, \tag{25}$$

$$u''(x) = \frac{e^x}{1 - e}. \tag{26}$$

First, we calculate the point $x^* = \ln(1/(e - 1))$, where $u'(x^*) = 0$ and verify that $u(x)$ attains its maximum value at x^* (since $u''(x^*) = -1/(e - 1)^2 < 0$). Since $u(0) = u(1) = 0$ we can conclude that the exact solution is monotonically increasing on $[0, x^*]$ and monotonically decreasing on $[x^*, 1]$. Since the numerical solution should have the same characteristics as the exact solution, oscillatory solutions should be considered as not reflecting the analytic solution.

(d) Next, we use $\Delta x = 1/4$, then, from equations (21) and (23), one obtains the following system

$$32w_1 - 14w_2 = 1 \tag{27}$$

$$-18w_1 + 32w_2 - 14w_3 = 1 \tag{28}$$

$$-18w_2 + 32w_3 = 1 \tag{29}$$

(e) The iteration process is a fixed point method. If the process converges we have: $\lim_{n\to\infty} x_n = p$. Using this in the iteration process yields:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} [x_n + h(x_n)(x_n^3 - 27)]$$

Since h is a continuous function one obtains:

$$p = p + h(p)(p^3 - 27)$$

so

$$h(p)(p^3 - 27) = 0.$$

Since $h(x) \neq 0$ for each $x \neq 0$ it follows that $p^3 - 27 = 0$ and thus $p = 27^{\frac{1}{3}} = 3$.

(f) The convergence of a fixed point method $x_{n+1} = g(x_n)$ is determined by $g'(p)$. If $|g'(p)| < 1$ the method converges, whereas if $|g'(p)| > 1$ the method diverges. For all choices we compute the first derivative in p. For the first method we elaborate all steps. For the other methods we only give the final result. For h_1 we have $g_1(x) = x - \frac{x^3-27}{x^4}$. The first derivative is:

$$g_1'(x) = 1 - \frac{3x^2 \cdot x^4 - (x^3 - 27) \cdot 4x^3}{(x^4)^2}$$

Substitution of p yields:

$$g_1'(p) = 1 - \frac{3p^6 - (p^3 - 27) \cdot 4p^3}{p^8}.$$

Since $p = 3$ the final term cancels:

$$g_1'(p) = 1 - \frac{3p^6}{p^8} = 1 - 3^{-1} = \frac{2}{3}.$$

This implies that the method is convergent with convergence factor $\frac{2}{3}$.

For the second method we have:

$$g_2'(p) = 1 - \frac{3p^4 - (p^3 - 27) \cdot 2p}{p^4} = 1 - \frac{3p^4}{p^4} = -2$$

Thus the method diverges.

For the third method we have:

$$g_3'(p) = 1 - \frac{9p^4 - (p^3 - 27) \cdot 6p}{9p^4} = 1 - \frac{9p^4}{9p^4} = 0$$

Thus the method is convergent with convergence factor 0.

Concluding we note that the third method is the fastest.

(g) To estimate the error in p we first approximate the function f in the neighboorhood of p by the first order Taylor polynomial:

$$P_1(x) = f(p) + (x - p)f'(p) = (x - p)f'(p).$$

Due to the measurement errors we know that

$$(x - p)f'(p) - \epsilon_{max} \leq \hat{P}_1(x) \leq (x - p)f'(p) + \epsilon_{max}.$$

This implies that the perturbed root $\hat{p}$ is bounded by the roots of $(x-p)f'(p) - \epsilon_{max}$ and $(x - p)f'(p) + \epsilon_{max}$, which leads to

$$p - \frac{\epsilon_{max}}{|f'(p)|} \leq \hat{p} \leq p + \frac{\epsilon_{max}}{|f'(p)|}.$$