DELFT UNIVERSITY OF TECHNOLOGY

FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400)
Thursday August 17th 2017, 18:30-21:30

1. (a) The local truncation error is defined as

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}, \tag{1}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + \Delta t \left(a_1 f(t_n, y_n) + a_2 f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n)) \right). \tag{2}$$

A Taylor expansion of f around (t_n, y_n) yields

$$f(t_n+\Delta t, y_n+\Delta t f(t_n, y_n)) = f(t_n, y_n)+\Delta t\frac{\partial f}{\partial t}(t_n, y_n)+\Delta t f(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)+O(\Delta t^2). \tag{3}$$

This is substituted into equation (2) to obtain

$$z_{n+1} = y_n+\Delta t \left(a_1 f(t_n, y_n) + a_2 \left[f(t_n, y_n) + \Delta t\frac{\partial f}{\partial t}(t_n, y_n) + \Delta t f(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n) \right] \right)+O(\Delta t^3). \tag{4}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{5}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} y'(t_n) \tag{6}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n) \tag{7}$$

Substituting these expressions into (4) shows that

$$z_{n+1} = y_n + \Delta t(a_1 + a_2)y'(t_n) + \Delta t^2 a_2 y''(t_n) + O(\Delta t^3). \tag{8}$$

A Taylor series for $y(t)$ around t_n gives for y_{n+1}

$$y_{n+1} = y(t_n + \Delta t) = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + O(\Delta t^3). \tag{9}$$

Equations (9) and (8) are substituted into relation (1) to obtain

$$\tau_{n+1}(\Delta t) = y'(t_n)(1-(a_1+a_2)) + \Delta t y''(t_n)\left(\frac{1}{2} - a_2\right) + O(\Delta t^2) \tag{10}$$

Hence

(a) $a_1 + a_2 = 1$ implies $\tau_{n+1}(\Delta t) = O(\Delta t)$;

(b) $a_1 + a_2 = 1$ and $a_2 = \frac{1}{2}$, that is, $a_1 = a_2 = \frac{1}{2}$, gives $\tau_{n+1}(\Delta t) = O(\Delta t^2)$.

(b) The test equation is given by

$$y' = \lambda y. \tag{11}$$

Application of the predictor step to the test equation gives

$$w^*_{n+1} = w_n + \lambda \Delta t w_n = (1+\lambda\Delta t)w_n. \tag{12}$$

The corrector step yields

$$w_{n+1} = w_n + \Delta t\,(a_1\lambda w_n + a_2\lambda(1+\lambda\Delta t)w_n) = (1+(a_1+a_2)\lambda\Delta t + a_2\lambda^2\Delta t^2)w_n. \tag{13}$$

Hence the amplification factor is given by

$$Q(\Delta t\lambda) = 1 + (a_1+a_2)\lambda\Delta t + a_2(\lambda\Delta t)^2. \tag{14}$$

(c) Let $\lambda < 0$ (so λ is real), then, for stability, the amplification factor must satisfy

$$-1 \le Q(\lambda\Delta t) \le 1, \tag{15}$$

from the previous assignment, we have

$$-1 \le 1 + (a_1+a_2)\lambda\Delta t + a_2(\lambda\Delta t)^2 \le 1 \Leftrightarrow -2 \le (a_1+a_2)\lambda\Delta t + a_2(\lambda\Delta t)^2 \le 0. \tag{16}$$

First, we consider the left inequality:

$$a_2(\lambda\Delta t)^2 + (a_1+a_2)\lambda\Delta t + 2 \ge 0 \tag{17}$$

For $\lambda\Delta t = 0$, the above inequality is satisfied, further the discriminant is given by $(a_1+a_2)^2 - 8a_2 < 0$. Here the last inequality follows from the given hypothesis. Hence the left inequality in relation (16) is always satisfied. Next we consider the right hand inequality of relation (16)

$$a_2(\lambda\Delta t)^2 + (a_1+a_2)\lambda\Delta t \le 0. \tag{18}$$

This relation is rearranged into

$$a_2(\lambda\Delta t)^2 \le -(a_1+a_2)\lambda\Delta t, \tag{19}$$

hence

$$a_2|\lambda\Delta t|^2 \le (a_1+a_2)|\lambda\Delta t| \Leftrightarrow |\lambda\Delta t| \le \frac{a_1+a_2}{a_2}, \qquad a_2 \ne 0. \tag{20}$$

This results into the following condition for stability

$$\Delta t \le \frac{a_1+a_2}{a_2|\lambda|}, \qquad a_2 \ne 0. \tag{21}$$

(d) We use the method with $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$ and $\Delta t = \frac{1}{2}$. Let

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \quad \underline{w}^1 = \begin{pmatrix} w_1^1 \\ w_2^1 \end{pmatrix}, \quad \underline{w}^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{22}$$

where the subscript stands for the component, whereas the superscript denotes the time–index. First, we carry out the prediction step

$$\underline{\hat{w}}^1 = \underline{w}^0 + \Delta t A \underline{w}^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ \\ -2 \end{pmatrix}. \tag{23}$$

Subsequently, we perform the corrector step

$$\underline{w}^1 = \underline{w}^0 + \frac{\Delta t}{2} \left(A\underline{w}^0 + A\underline{\hat{w}}_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \tag{24}$$

Using $\Delta t = \frac{1}{2}$, gives

$$\underline{w}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4} \left(\begin{pmatrix} 0 \\ -4 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2} \\ \\ -1\frac{3}{4} \end{pmatrix}. \tag{25}$$

(e) Before we can investigate the stability of the method we first have to determine the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \tag{26}$$

It is easy to see that this matrix has two complex eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. It is sufficient to investigate if $|Q(\lambda_1 \Delta t)| \leq 1$ because $|Q(\lambda_2 \Delta t)| = |Q(\lambda_1 \Delta t)|$. Using $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$ and $\lambda_1 = 2i$ we obtain the following expression for $Q(\lambda_1 \Delta t)$

$$Q(\lambda_1 \Delta t) = 1 + \lambda_1 \Delta t + \frac{1}{2}(\lambda_1 \Delta t)^2 = 1 + 2\Delta t i - 2\Delta t^2$$

Substituting $\Delta t = \frac{1}{2}$ gives:

$$Q(\lambda_1 \Delta t) = 1 + i - \frac{1}{2}$$

Since $|Q(\lambda_1 \Delta t)| = \sqrt{(\frac{1}{2})^2 + 1^2)} = \sqrt{\frac{5}{4}} > 1$ the method is unstable for $\Delta t = \frac{1}{2}$.

2. (a) The **linear Lagrangian interpolatory polynomial**, with nodes x_0 and x_1, is given by

$$L_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1). \tag{27}$$

This is evident from application of the given formula.

(b) The **quadratic Lagrangian interpolatory polynomial** with nodes x_0, x_1 and x_2 is given by

$$\begin{aligned} L_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) && (28) \\ &+ \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) && (29) \\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2). && (30) \end{aligned}$$

This is also evident from application of the given formula.

(c) Obviously, $L_1(3) = 6$ and $L_2(3) = 6$ since the Lagrange interpolation polynomial satisfies $L_n(x_k) = f(x_k)$ for all points $x_0, x_1, \ldots, x_n$. Next, we compute $L_1(2)$ and $L_2(2)$ for both linear and quadratic Lagrangian interpolation as approximations at $x = 3$. For **linear interpolation**, we have

$$L_1(3) = \frac{2-3}{1-3} \cdot 3 + \frac{2-1}{3-1} \cdot 6 = \frac{9}{2}, \tag{31}$$

and for **quadratic interpolation**, one obtains

$$L_2(3) = \frac{(2-3)(2-4)}{(1-3)(1-4)} \cdot 3 + \frac{(2-1)(2-4)}{(3-1)(3-4)} \cdot 6 + \frac{(2-1)(2-3)}{(4-1)(4-3)} \cdot 5 = \frac{16}{3}. \tag{32}$$

3. (a) Consider an interval of integration $[x_{j-1}, x_j]$, then the **Rectangle Rule** reads

$$I_j^R = hf(x_{j-1}), \qquad h = x_j - x_{j-1}. \tag{33}$$

The *composed integration rule* is derived by

$$I^R = h(I_1^R + I_2^R + \ldots + I_n^R) = h(f(x_0) + \ldots + f(x_{n-1})), \tag{34}$$

which yields

$$I^R(h = 1/4) = \frac{1}{4} \cdot (0 + (\frac{1}{4})^2 + (\frac{2}{4})^2 + (\frac{3}{4})^2) = \frac{7}{32}. \tag{35}$$

(b) For the interval of integration $[x_{j-1}, x_j]$ the **Trapezoidal Rule** is

$$I_j^T = \frac{h}{2}(f(x_{j-1}) + f(x_j)). \tag{36}$$

The *composed integration rule* is derived by

$$I^T = h(I_1^T + I_2^T + \ldots + I_n^T) = h(\frac{f(x_0)}{2} + f(x_1) + \ldots + f(x_{n-1}) + \frac{f(x_n)}{2}), \tag{37}$$

which leads to

$$I^T(h = 1/4) = \frac{1}{4} \cdot (0 + (\frac{1}{4})^2 + (\frac{2}{4})^2 + (\frac{3}{4})^2 + \frac{1}{2}) = \frac{11}{32}. \tag{38}$$

(c) For a general number of subintervals, say n, the magnitude of the composed Rectangle- and Trapezoidal Rules, is bounded from above by

$$\begin{aligned} \varepsilon_R &\leq \frac{h}{2} \max_{x\in[0,1]} |y'(x)| \leq h = \frac{1}{n}, \\ \varepsilon_T &\leq \frac{h^2}{12} \max_{x\in[0,1]} |y''(x)| \leq \frac{h^2}{6} = \frac{1}{6n^2}. \end{aligned} \tag{39}$$

Here, the exact solution $y(x) = x^2$ was used. Hence, the error from the Trapezoidal Rule is much smaller. Furthermore, from the composed Rules, it is easy to see that the number of function evaluations for the composed Rectangle- and Trapezoidal Rules is given by n and $n + 1$, respectively. Since

$$\lim_{n\to\infty} \frac{n+1}{n} = 1, \tag{40}$$

it follows that the amount of work for the Trapezoidal Rule is not significantly higher than it is for the Rectangle Rule. Hence, it is more attractive to use the Trapezoidal Rule.

(d) It follows from the given error formula that

$$\int_0^1 y(x)dx - I^T(h) = c_p h^p \tag{41}$$

$$\int_0^1 y(x)dx - I^T(2h) = c_p (2h)^p \tag{42}$$

$$\int_0^1 y(x)dx - I^T(4h) = c_p (4h)^p \tag{43}$$

By subtracting equation (42) from (43) and equation (41) from (42), the unknown exact integral value can be eliminated

$$I^T(2h) - I^T(4h) = c_p(2h)^p(2^p - 1) \tag{44}$$

$$I^T(h) - I^T(2h) = c_p(h)^p(2^p - 1) \tag{45}$$

By dividing these two expressions we obtain

$$\frac{I^T(2h) - I^T(4h)}{I^T(h) - I^T(2h)} = 2^p \tag{46}$$

From part (b) we know that $I^T(h = 1/4) = 11/32$. Moreover,

$$I^T(h = 1/2) = \frac{1}{2} \cdot (0 + (\frac{1}{2})^2 + \frac{1}{2}) = \frac{3}{8}. \tag{47}$$

and

$$I^T(h=1) = 1 \cdot (0 + \frac{1}{2}) = \frac{1}{2}. \tag{48}$$

Filling these three values into the above error formula yields

$$\frac{\frac{3}{8} - \frac{1}{2}}{\frac{11}{32} - \frac{3}{8}} = 4 = 2^p \tag{49}$$

from which it follows that $p = 2$.

(e) By dividing equation (45) by $(2^p - 1)$ we obtain

$$\frac{I^T(h) - I^T(2h)}{2^p - 1} = c_p(h)^p \tag{50}$$

which, according to the given error formula (41) equals

$$\frac{I^T(h) - I^T(2h)}{2^p - 1} = c_p(h)^p = \int_0^1 y(x)dx - I^T(h) \tag{51}$$

It follows that the error of the Trapozoidal Rule for $h = 1/4$ can be estimated as

$$\frac{\frac{11}{32} - \frac{3}{8}}{4 - 1} = -0.01042 \tag{52}$$