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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday February 1st 2018, 18:30-21:30

1. (a) Consider the test equation $y' = \lambda y$, then it follows that

$$\begin{aligned}
k_1 &= \lambda \Delta t w_n && (1)\\
k_2 &= \lambda \Delta t \left(w_n + \frac{1}{2} \lambda \Delta t w_n \right) && (2)\\
&= \left(\lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2 \right) w_n && (3)\\
k_3 &= \lambda \Delta t \left(w_n - \lambda \Delta t w_n + 2 \left(\lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2 \right) w_n \right) && (4)\\
&= \left(\lambda \Delta t + (\lambda \Delta t)^2 + (\lambda \Delta t)^3 \right) w_n && (5)\\
w_{n+1} &= w_n + \alpha \lambda \Delta t w_n + \beta \left(\lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2 \right) w_n && (6)\\
&\quad + \gamma \left(\lambda \Delta t + (\lambda \Delta t)^2 + (\lambda \Delta t)^3 \right) w_n && (7)\\
&= \left(1 + (\alpha + \beta + \gamma) \lambda \Delta t + \left(\frac{1}{2} \beta + \gamma \right) (\lambda \Delta t)^2 + \gamma (\lambda \Delta t)^3 \right) w_n && (8)
\end{aligned}$$

Hence the amplification factor is given by

$$Q(\lambda \Delta t) = 1 + (\alpha + \beta + \gamma) \lambda \Delta t + \left(\frac{1}{2} \beta + \gamma \right) (\lambda \Delta t)^2 + \gamma (\lambda \Delta t)^3 . \quad (9)$$

(b) The local truncation error for the test equation $y' = \lambda y$ is given by

$$\tau_{n+1}(\Delta t) = \frac{e^{\lambda \Delta t} - Q(\lambda \Delta t)}{\Delta t} y_n. \quad (10)$$

The Taylor Series around 0 for $e^{\lambda \Delta t}$ is:

$$e^{\lambda \Delta t} = 1 + \lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2 + \frac{1}{6} (\lambda \Delta t)^3 + \mathcal{O}(\Delta t^4). \quad (11)$$

Hence, this gives

$$\begin{aligned}
e^{\lambda \Delta t} - Q(\lambda \Delta t) &= (1 - \alpha - \beta - \gamma) \lambda \Delta t + \left(\frac{1}{2} - \frac{1}{2} \beta - \gamma \right) (\lambda \Delta t)^2 && (12)\\
&\quad + \left(\frac{1}{6} - \gamma \right) (\lambda \Delta t)^3 + \mathcal{O}(\Delta t^4). && (13)
\end{aligned}$$

and hence $\tau_{n+1}(\Delta t) = \mathcal{O}(\Delta t^3)$ only if

$$\begin{aligned} \alpha + \beta + \gamma &= 1, && (14) \\ \frac{1}{2}\beta + \gamma &= \frac{1}{2}, && (15) \\ \gamma &= \frac{1}{6}, && (16) \\ & && (17) \end{aligned}$$

which have as solution

$$\begin{aligned} \alpha &= \frac{1}{6}, && (18) \\ \beta &= \frac{2}{3}, && (19) \\ \gamma &= \frac{1}{6}. && (20) \\ & && (21) \end{aligned}$$

(c) Let $x_1 = y$ and $x_2 = y'$, then it follows that $y'' = x_2'$, and hence we get

$$\begin{array}{l} x_2' + x_2 + \frac{1}{2}x_1 = t, \\ x_2 = x_1'. \end{array} \qquad (22)$$

This expression is written as

$$\begin{array}{l} x_1' = x_2, \\ x_2' = -\frac{1}{2}x_1 - x_2 + t. \end{array} \qquad (23)$$

Finally, we get the following matrix–form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix}. \qquad (24)$$

Here, we have $A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}$ and $f = \begin{bmatrix} 0 \\ t \end{bmatrix}$. The initial conditions are given by $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) To this extent, we determine the eigenvalues of the matrix A. Subsequently, these eigenvalues are substituted into the amplification factor. The eigenvalues of A are given by $-\frac{1}{2} \pm \frac{1}{2}i$. Using $\Delta t = 2$, it follows that

$$\begin{aligned} Q(\lambda \Delta t) &= 1 + \lambda \Delta t + \frac{1}{2}\lambda^2 \Delta t^2 + \frac{1}{6}\lambda^3 \Delta t^3 && (25) \\ &= 1 + (-1+i) + \frac{1}{2}(-1+i)^2 + \frac{1}{6}(-1+i)^3 && (26) \\ &= \frac{1}{3} - \frac{1}{3}i. && (27) \end{aligned}$$

Herewith, it follows that $|Q(\lambda\Delta t)|^2 = \frac{2}{9} < 1$. Hence for $\Delta t = 2$, it follows that the method applied to the given system is stable. Note that this conclusion holds for both the eigenvalues of A since they are complex conjugates.

(e) The given method, applied to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{cases} \underline{k}_1 = \Delta t \left(A\underline{w}_n + \underline{f}\,(t_n)\right) \\ \underline{k}_2 = \Delta t \left(A\left(\underline{w}_n + \frac{1}{2}\underline{k}_1\right) + \underline{f}\left(t_n + \frac{1}{2}\Delta t\right)\right) \\ \underline{k}_3 = \Delta t \left(A\left(\underline{w}_n - \underline{k}_1 + 2\underline{k}_2\right) + \underline{f}\,(t_n + \Delta t)\right) \\ \underline{w}_{n+1} = \underline{w}_n + \frac{1}{6}\left(\underline{k}_1 + 4\underline{k}_2 + \underline{k}_3\right) \end{cases} \tag{28}$$

With the initial condition and $\Delta t = 2$, this gives

$$\begin{cases} \underline{k}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ \underline{k}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \underline{k}_3 = \begin{bmatrix} 12 \\ -5 \end{bmatrix} \\ \underline{w}_1 = \begin{bmatrix} 8/3 \\ 1/3 \end{bmatrix} \end{cases} \tag{29}$$

2. (a) The first order backward difference formula for the first derivative is given by

$$d'(t) \approx \frac{d(t) - d(t-h)}{h}.$$

Using $t = 20$, and $h = 10$ the approximation of the velocity is

$$\frac{d(20) - d(10)}{10} = \frac{100 - 40}{10} = 6 \text{ (m/s)}.$$

(b) Taylor polynomials are:

$$\begin{aligned} d(0) &= d(2h) - 2hd'(2h) + 2h^2 d''(2h) - \frac{(2h)^3}{6} d'''(\xi_0)\,, \\ d(h) &= d(2h) - hd'(2h) + \frac{h^2}{2} d''(2h) - \frac{h^3}{6} d'''(\xi_1)\,, \\ d(2h) &= d(2h). \end{aligned}$$

We know that $Q(h) = \frac{\alpha_0}{h}d(0) + \frac{\alpha_1}{h}d(h) + \frac{\alpha_2}{h}d(2h)$, which should be equal to $d'(2h) + O(h^2)$. This leads to the following conditions:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\ , \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\ , \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\ . \end{array}$$

(c) The truncation error follows from the Taylor polynomials:

$$d'(2h) - Q(h) = d'(2h) - \frac{d(0) - 4d(h) + 3d(2h)}{2h} = \frac{\frac{8h^3}{6}d'''(\xi_0) - 4(\frac{h^3}{6}d'''(\xi_1))}{2h} = \frac{1}{3}h^2 d'''(\xi).$$

(d) Using the new formula with $h = 10$ we obtain the estimate:

$$\frac{d(0) - 4d(10) + 3d(20)}{20} = \frac{0 - 4 \times 40 + 3 \times 100}{20} = 7 \text{ (m/s)}.$$

3. (a) Newton–Raphson's Method is an iterative method to find $p \in \mathbb{R}$ such that $f(p) = 0$. One constructs a sequence of successive approximations $\{p_n\}$. Given the n–th estimate, then p_{n+1} is obtained through linearizing around p_n and by finding p_{n+1} by determining the point where the linearization (tangent) equals zero. Linearization of $f(p)$ around p_n gives (upon neglecting the error)

$$f(p) \approx f(p_n) + f'(p_n)(p - p_n) =: L(p; p_n), \tag{30}$$

for any p provided the second derivative of $f(p)$ is bounded and where $L(p; p_n)$ denotes the tangent (linearization) of $f(p)$ at point $(p_n, f(p_n))$. Then the next point is found upon setting $L(p_{n+1}; p_n) = 0$:

$$f(p_n) + f'(p_n)(p_{n+1} - p_n) = 0. \tag{31}$$

The above equation is solved for p_{n+1}, and gives

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \tag{32}$$

which is the famous Newton–Raphson formula for root–finding. For the graphical derivation, see Figure 4.2 in the book.

(b) The Jacobian matrix of $\mathbf{f}(\mathbf{x})$ is defined by

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_1}{\partial x_m}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_m}{\partial x_m}(\mathbf{x}) \end{pmatrix}.$$

The definition of the Newton–Raphson method is:

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n-1)} - \mathbf{J}^{-1}(\mathbf{p}^{(n-1)})\mathbf{f}(\mathbf{p}^{(n-1)}). \tag{33}$$

(c) First, we rewrite the system into the form

$$\begin{aligned} f_1(p_1, p_2) &= 0, \\ f_2(p_1, p_2) &= 0, \end{aligned} \tag{34}$$

by setting

$$\begin{aligned} f_1(p_1, p_2) &:= 18p_1 - 9p_2 + (p_1)^2, \\ f_2(p_1, p_2) &:= -9p_1 + 18p_2 + (p_2)^2 - 9. \end{aligned} \tag{35}$$

We denote the Jacobian matrix by $\mathbf{J}(p_1, p_2)$. Note that

$$\mathbf{J}(\mathbf{p}) = \begin{pmatrix} 18 + 2p_1^{(0)} & -9 \\ -9 & 18 + 2p_2^{(0)} \end{pmatrix}. \tag{36}$$

Using $p_1^{(0)} = p_2^{(0)} = 0$ we obtain:

$$\mathbf{J}^{-1}(\mathbf{p}^{(0)}) = \begin{pmatrix} 18 & -9 \\ -9 & 18 \end{pmatrix}. \tag{37}$$

This implies that

$$\mathbf{J}^{-1}(\mathbf{p}^{(0)})^{-1} = \frac{1}{18^2 - 81} \begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix}. \tag{38}$$

Furthermore

$$\mathbf{f}(\mathbf{p}^{(0)}) = \begin{pmatrix} 0 \\ -9 \end{pmatrix}, \tag{39}$$

so

$$\mathbf{p}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{18^2 - 81} \begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}. \tag{40}$$