DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3097TU WI3097Minor WI3197Minor AESB2210 AESB2210-18 CTB2400)
Tuesday April 16th 2019, 13:30-16:30

Number of questions: This is an exam with 12 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $^P/_2$, where P is the number of points earned.

1. We consider the following time-integration method

$$w_{n+1} = w_n + \frac{1}{2}\Delta t \left(f(t_n, w_n) + f(t_{n+1}, w_n + \Delta t f(t_n, w_n)) \right) \tag{1}$$

for the integration of the **initial value problem** $y' = f(t, y)$, $y(t_0) = y_0$.

(a) Show that the *local truncation error* of (1) applied to $y' = g(y)$ takes on the form

$$\tau_{n+1} = P\Delta t^2 + \mathcal{O}(\Delta t^3),$$

and *give* a formula for P. (4 pt.)
Hint: $y'' = g'(y)y'$.
Remark: Usage of the test equation in question (a) will result in zero points for question (a).

(b) Demonstrate that the *amplification factor* of (1) is given by

$$Q(\lambda\Delta t) = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2.$$

(1 pt.)

(c) We consider the following *system of linear differential equations*:

$$\begin{cases} x_1' &= -\frac{3}{2}x_1 - \frac{1}{2}x_2 - 1, \\ x_2' &= -\frac{1}{2}x_1 - \frac{3}{2}x_2 + 1, \\ x_1(0) &= 0, \\ x_2(0) &= 0. \end{cases} \tag{2}$$

Write the above system as $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ and *derive* the most strict upper bound $\Delta t_{\max}$ for Δt such that the application of (1) to (2) is stable for $\Delta t \leq \Delta t_{\max}$. (4 pt.)
Hint: The eigenvalues of A are real numbers.
Remark: Wrong eigenvalues will lead to a substraction of 1 point in question (c).

(d) *Perform* one time step by applying (1) to (2) with $\Delta t = 1$. (1 pt.)
Remark: Using a different method or solving a different system will result in zero points for question (d).

2. We consider the following convection-diffusion boundary-value problem:

$$\begin{cases} -y''(x) - 3y'(x) &= 1, \quad x \in (0,1], \\ y'(0) &= 0, \\ y(1) &= 1. \end{cases} \tag{3}$$

In this exercise we approximate the exact solution with a numerical method.

(a) The derivative $y'(x)$ will be approximated by an upwind discretization $U(\Delta x)$, for which there are two candidates available:

$$U(\Delta x) = \frac{y(x) - y(x - \Delta x)}{\Delta x}, \tag{4}$$

or

$$U(\Delta x) = \frac{y(x + \Delta x) - y(x)}{\Delta x}. \tag{5}$$

Argue which candidate should be used and *determine* the order of this approximation. ($1\frac{1}{2}$ pt.)

(b) *Derive* an $\mathcal{O}(\Delta x^2)$ approximation $Q(\Delta x)$ for $y''(x)$ which is of the form

$$Q(\Delta x) = \frac{\alpha_1 y(x + \Delta x) + \alpha_0 y(x) + \alpha_{-1} y(x - \Delta x)}{\Delta x^2}.$$

(2 pt.)

(c) We solve the boundary value problem (3) using the finite differences $Q(\Delta x)$ and $U(\Delta x)$, after setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, with Δx as the uniform step size. *Derive* the resulting scheme, including arguments, for an arbitrary internal node x_j and for all boundary nodes. ($2\frac{1}{2}$ pt.)
Remark: Your choice for $U(\Delta x)$ in question (a) and for $Q(\Delta x)$ in question (b) will not influence the points for question (c), if applied correctly.

3. We want to find an approximation of $\sqrt{3}$. Therefore we consider the fixed-point problem

$$x = g(x),$$

on the interval $[1, 2]$, where the function g is defined as

$$g(x) = -\frac{1}{3}x^2 + x + 1. \tag{6}$$

In the next exercises you will prove that $p = \sqrt{3}$ is indeed the unique fixed point of g in the interval $[1, 2]$ and that for any starting value of $p_0 \in [1, 2]$ the fixed-point iteration

$$p_{n+1} = g(p_n), \tag{7}$$

converges to $p = \sqrt{3}$, and you will perform this fixed-point iteration.

(a) *Show* that $p = \sqrt{3}$ is a fixed point of the function g. ($\frac{1}{2}$ pt.)

(b) *Argue* why g is continuous on $[1, 2]$. ($\frac{1}{2}$ pt.)

(c) *Show* that $1 \leq g(x) \leq 2$ for all $x \in [1, 2]$. (1 pt.)

(d) *Find* the smallest value k such that $|g'(x)| \leq k < 1$ for all $x \in [1, 2]$. (1 pt.)

(e) *Approximate* $p = \sqrt{3}$ by calculating p_1 and p_2 with 4 significant digits, given that $p_0 = 2.000$. (1 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html