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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3097TU WI3197Minor AESB2210-18 CTB2400)
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1. (a) The local truncation error is given by

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}, \tag{1}$$

with $y_{n+1} = y(t_{n+1})$ the exact solution at time t_{n+1} and
z_{n+1} the numerical approximation obtained with $w_n = y_n$.
y_{n+1} can be expanded by the use of Taylor expansions around t_n:

$$y_{n+1} = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \mathcal{O}(\Delta t^3). \tag{2}$$

After substitution of the predictor $z^*_{n+1} = y_n + \Delta t f(t_n, y_n)$ into the corrector, and after using a Taylor expansion around (t_n, y_n), we obtain for z_{n+1}:

$$\begin{aligned} z_{n+1} &= y_n + \frac{\Delta t}{2}\left(f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))\right), \\ &= y_n + \frac{\Delta t}{2}\left(2f(t_n, y_n) + \Delta t \left(\frac{\partial f(t_n, y_n)}{\partial t} + y'(t_n)\frac{\partial f(t_n, y_n)}{\partial y}\right) + \mathcal{O}(\Delta t^2)\right), \\ &= y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \mathcal{O}(\Delta t^3). \end{aligned}$$

So we obtain

$$y_{n+1} - z_{n+1} = \mathcal{O}(\Delta t^3), \text{ and hence } \tau_{n+1}(\Delta t) = \frac{\mathcal{O}(\Delta t^3)}{\Delta t} = \mathcal{O}(\Delta t^2). \tag{3}$$

(b) Application of the integration method to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{aligned} \underline{w}_1^* &= \underline{w}_0 + \Delta t \left(A\underline{w}_0 + \underline{f}_0 \right), \\ \underline{w}_1 &= \underline{w}_0 + \tfrac{\Delta t}{2} \left(A\underline{w}_0 + f_0 + A\underline{w}_1^* + \underline{f}_1 \right). \end{aligned} \tag{4}$$

With the initial condition $\underline{w}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\Delta t = 1$, this gives the following result for the predictor

$$\begin{aligned} \underline{w}_1^* &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \left(\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

The corrector is calculated as follows

$$\begin{aligned} \underline{w}_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2} \left(\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(1) \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(1) \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 + \sin(1) \end{pmatrix} \\ &= \begin{pmatrix} 1/2 \\ 1/2 + 1/2 \sin(1) \end{pmatrix}. \end{aligned}$$

(c) The amplification factor $Q(\lambda\Delta t)$ is defined as

$$Q(\lambda\Delta t) = \frac{w_{n+1}}{w_n},$$

with w_{n+1} de result of applying the given method to the test equation $y' = \lambda y$. Applying the method results in:

$$\begin{aligned} w^*_{n+1} &= w_n + \Delta t\,(\lambda w_n) \\ &= (1 + \lambda\Delta t)\,w_n, \end{aligned}$$

and

$$\begin{aligned} w_{n+1} &= w_n + \frac{\Delta t}{2}\left(\lambda w_n + \lambda\left((1 + \lambda\Delta t)\,w_n\right)\right) \\ &= w_n + \frac{\Delta t}{2}\left(\lambda w_n + \left(\lambda + \lambda^2\Delta t\right)w_n\right) \\ &= w_n + \frac{\Delta t}{2}\left(2\lambda w_n + \lambda^2\Delta t w_n\right) \\ &= w_n + \lambda\Delta t w_n + \frac{1}{2}\lambda^2\Delta t^2 w_n \\ &= \left(1 + \lambda\Delta t + \frac{1}{2}\lambda^2\Delta t^2\right)w_n. \end{aligned}$$

This finally leads to

$$Q(\lambda\Delta t) = 1 + \lambda\Delta t + \frac{1}{2}\lambda^2\Delta t^2.$$

(d) For stability,

$$|Q(\lambda\Delta t)| \leq 1,$$

must hold for all eigenvalues of the linear initial value problem, with Q the amplification factor of the given method.

First, we determine the eigenvalues of the matrix A. Subsequently, the eigenvalues are substituted into the amplification factor.

The eigenvalues of the matrix A are given by $\lambda_1 = 0$ and $\lambda_2 = -1$.

We first consider $\lambda_1 = 0$:

$$\begin{aligned} Q(\lambda_1\Delta t) &= 1 + 0\Delta t + \frac{1}{2}(0\Delta t)^2 \\ &= 1. \end{aligned}$$

From this it easily follows that

$$|Q(\lambda_1\Delta t)| \leq 1,$$

and therefore $\lambda_1 = 0$ sets no restrictions on the value of Δt.

Now we consider $\lambda_2 = -1$:

$$\begin{aligned} Q(\lambda_2\Delta t) &= 1 + (-1)\Delta t + \frac{1}{2}(-1\Delta t)^2 \\ &= 1 - \Delta t + \frac{1}{2}\Delta t^2. \end{aligned}$$

From this it follows that Δt should satisfy

$$-1 \leq 1 - \Delta t + \frac{1}{2}\Delta t^2 \leq 1.$$

Consider the left inequality, which can be rewritten to:

$$\frac{1}{2}\Delta t^2 - \Delta t + 2 \geq 0.$$

This inequality is satisfied if the quadratic function on the left has no real roots and there is one value of Δt such that the inequality is satisfied.

Substituting $\Delta t = 1$ ($\Delta t = 0$ cannot be taken, as $\Delta t > 0$ is given in the exercise) gives

$$3/2 \geq 0,$$

which is true. The discriminant of the function is given by

$$D = (-1)^2 - 4 \cdot \frac{1}{2} \cdot 2 = -3 < 0$$

so the quadratic function has no real roots. Therefore the left inequality sets no restrictions on the value of Δt.

Consider the right inequality, which can be rewritten as:

$$\begin{aligned} 1 - \Delta t + \frac{1}{2}\Delta t^2 &\leq 1 \\ -\Delta t + \frac{1}{2}\Delta t^2 &\leq 0 \\ -1 + \frac{1}{2}\Delta t &\leq 0 \quad \text{because } \Delta t > 0 \text{ is given.} \\ \frac{1}{2}\Delta t &\leq 1 \\ \Delta t &\leq 2. \end{aligned}$$

From the above it follows that the method applied to the initial value problem is stable if

$$\Delta t \leq 2.$$

2. (a) Evaluation of the ode in $x = x_j$ and replacing $y''(x_j)$ with a finite difference of $\mathcal{O}(\Delta x^2)$ gives

$$-\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{\Delta x^2} + \mathcal{O}\left(\Delta x^2\right) + 4y(x_j) = 4e^{2j\Delta x}.$$

Next, we neglect the truncation error,
and set $w_j \approx y(x_j)$ to obtain

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} + 4w_j = 4e^{2j\Delta x}, \tag{5}$$

which is the second of the given equations.
At the left boundary, $x = 0$, we have $w_0 = \frac{3}{2}$, which after substitution in (5) for $j = 1$ gives

$$-\frac{w_2 - 2w_1}{\Delta x^2} + 4w_1 = 4e^{2\Delta x} + \frac{3}{2\Delta x^2},$$

which is the first of the given equations.
At the right boundary, $x = 1$, we approximate $y'(1)$ with a second-order central finite-difference, which transforms the boundary condition in:

$$\frac{y(x_{n+2}) - y(x_n)}{2\Delta x} + \mathcal{O}(\Delta x^2) = 0,$$

which after neglecting the errors results in

$$w_{n+2} = w_n.$$

Substitution of the above in (5) with $j = n + 1$ and division by two gives

$$-\frac{-w_{n+1} + w_n}{\Delta x^2} + 2w_{n+1} = 2e^2,$$

which is the third of the given equations.

(b) *Each mistake in an equation (directly stated A and* $\mathbf{b}$*) results in a subtraction of* $^1/_4$ *point, with at most the allocated points being subtracted.*

We use $\Delta x = 1/4$, so $n = 4$ and then, from the given equations, one obtains the following system:

$$\begin{aligned} 36w_1 - 16w_2 &= 4e^{1/2} + 24 \\ -16w_1 + 36w_2 - 16w_3 &= 4e \\ -16w_2 + 36w_3 - 16w_4 &= 4e^{3/2} \\ -16w_3 + 18w_4 &= 2e^2 \end{aligned}$$

This means with $\mathbf{w} = [w_1, w_2, w_3, w_4]^T$ that

$$A = \begin{bmatrix} 36 & -16 & 0 & 0 \\ -16 & 36 & -16 & 0 \\ 0 & -16 & 36 & -16 \\ 0 & 0 & -16 & 18 \end{bmatrix},$$

and

$$\mathbf{b} = \begin{bmatrix} 4e^{1/2} + 24 \\ 4e \\ 4e^{3/2} \\ 2e^2 \end{bmatrix}.$$

3. (a) Approximating $f(x)$ by $L_2(x)$ and integration over x from a to b gives:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b L_2(x)\,\mathrm{d}x \\
&= \int_a^b f(a)L_{02}(x) + f\left(\frac{a+b}{2}\right)L_{12}(x) + f(b)L_{22}(x)\,\mathrm{d}x \\
&= f(a)\int_a^b L_{02}(x)\,\mathrm{d}x + f\left(\frac{a+b}{2}\right)\int_a^b L_{12}(x)\,\mathrm{d}x + f(b)\int_a^b L_{22}(x)\,\mathrm{d}x \\
&= f(a)\frac{b-a}{6} + f\left(\frac{a+b}{2}\right)\frac{2(b-a)}{3} + f(b)\frac{b-a}{6} \\
&= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).
\end{aligned}$$

(b) Let f be an arbitrary polynomial of degree 3 or lower. So f must be of the form

$$f(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4,$$

with $c_i, i = 1, 2, 3, 4$ constants. But this means:

$$\begin{aligned}
 & & f'(x) &= 3c_1 x^2 + 2c_2 x + c_3, \\
\Rightarrow & & f''(x) &= 6c_1 x + 2c_2, \\
\Rightarrow & & f^{(3)}(x) &= 6c_1, \\
\Rightarrow & & f^{(4)}(x) &= 0, \\
\Rightarrow & & |f^{(4)}(x)| &= 0, \\
\Rightarrow & & \max_{a \le x \le b} |f^{(4)}(x)| &= 0, \\
\Rightarrow & & m_4 &= 0.
\end{aligned}$$

The given inequality for the truncation error therefore becomes

$$\left| \int_a^b f(x)\,\mathrm{d}x - I_S \right| \le 0,$$

which shows that the Simpon's rule is exact for polynomials of degree 3 and lower.

(c) Applying Simpson's rule with $a = 0$, $b = \pi$ and $f(x) = \sin(x)$ results in

$$\begin{aligned}\int_0^\pi \sin(x)\mathrm{d}x &\approx \frac{\pi}{6}\left(0 + 4 \cdot 1 + 0\right) \\ &= \frac{2\pi}{3}.\end{aligned}$$

We have $f^{(4)}(x) = \sin(x)$, so $m_4 = 1$, which gives

$$\left|\int_0^\pi \sin(x)\mathrm{d}x - I\right| \leq \frac{\pi^5}{2880}.$$