DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400) Thursday June 23 2022, 13:30-16:30

Number of questions: This is an exam with 12 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. For the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, we use the following integration method:

$$\begin{cases} w^*_{n+1} = w_n + \Delta t f(t_n, w_n) \\ w_{n+1} = w_n + \dfrac{\Delta t}{2}\left(f(t_n, w_n) + f(t_{n+1}, w^*_{n+1})\right). \end{cases} \tag{1}$$

Here Δt denotes the timestep and w_n represents the numerical approximation at time t_n.

(a) Show that the local truncation error of the integration method is of the order $\mathcal{O}(\Delta t^2)$. (You are not allowed to use the test equation here.) (3pt.)

Consider the following initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = \cos \pi t, \\ y(0) = 1, \quad \dfrac{dy}{dt}(0) = 0. \end{cases} \tag{2}$$

(b) Show that the above initial value problem can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos \pi t \end{pmatrix}. \tag{3}$$

Give the initial conditions for $x_1(0)$ and $x_2(0)$ as well. (1pt.)

(c) Calculate one step with integration method (1), in which $\Delta t = 0.5$ and $t_0 = 0$, applied to (3) and use the given initial conditions. (2pt.)

(d) Show that the amplification factor for this integration method is given by: $Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2}$. (2pt.)

(e) Examine for which stepsizes Δt, the integration method (1), applied to the initial value problem (3), is stable. (2pt.)

please turn over

2. In this exercise an estimate is determined for the velocity of a bike. The measured distances of the bike from the starting line are given in the table below.

t (s)	0	10	20
$d(t)$ (m)	0	40	100

   (a) Give the first order backward difference formula for $d'(2h)$ and use this to determine an estimate of the velocity for $t = 20$. (1 pt.)

   (b) We are looking for a difference formula of the first derivative of d in $2h$ of the form:

$$Q(h) = \frac{\alpha_0}{h}d(0) + \frac{\alpha_1}{h}d(h) + \frac{\alpha_2}{h}d(2h), \text{ such that } Q(h) - d'(2h) = O(h^2).$$

   In the remainder of this exercise we use this formula. Show that the coefficients α_0, α_1 and α_2 should satisfy the next system:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\,, \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\,, \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\,. \end{array}$$

   (2 pt.)

   (c) The solution of this system is given by $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$. Show that the truncation error can be written as: $Q(h) - d'(2h) = O(h^2)$. (1 pt.)

   (d) Use $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$ in $Q(h)$ to give an estimate of the velocity at $t = 20$. (1 pt.)

3. We derive and use **Newton-Raphson's method** to solve a nonlinear problem.

   (a) Given is the *scalar* nonlinear problem:

$$\text{Find } p \in \mathbb{R} \text{ such that } f(p) = 0. \qquad (4)$$

   Derive Newton–Raphson's formula (a graphical explanation is also allowed), given by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \text{ for } n \geq 1 \qquad (5)$$

   (2 pt.)

   (b) Derive Newton-Raphson's method for the *general* nonlinear problem:

$$\text{Find } \mathbf{p} \in \mathbb{R}^m \text{ such that } \mathbf{f}(\mathbf{p}) = \mathbf{0}. \qquad (6)$$

   (1 pt.)

   (c) Perform **one** step of Newton-Raphson's method applied to the following nonlinear problem for w_1 and w_2:

$$\begin{cases} 18w_1 - 9w_2 + w_1^2 = 0, \\ -9w_1 + 18w_2 + w_2^2 = 9. \end{cases} \qquad (7)$$

   Use $w_1 = w_2 = 0$ as the initial estimate. (2 pt.)

For the answers of this test we refer to:
http://homepage.tudelft.nl/d2b4e/wi3097/tentamen.html