DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

Examiner responsible: C. Vuik
Examination reviewer: D. den Ouden-van der Horst

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400) Tuesday July 18 2023, 13:30-16:30

Number of questions: This is an exam with 11 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. For the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, we use the following integration method:

$$\begin{cases} k_1 = f(t_n, w_n) \\ k_2 = f(t_{n+1}, w_n + \Delta t k_1) \\ w_{n+1} = w_n + \dfrac{\Delta t}{2}(k_1 + k_2), \end{cases} \tag{1}$$

Here Δt denotes the time step and w_n represents the numerical approximation at time t_n.

(a) Show that the local truncation error of the integration method is of the order $\mathcal{O}(\Delta t^2)$. *(You are not allowed to use the test equation here.)* (3pt.)

(b) Consider the following initial value problem

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos \pi t \end{pmatrix} \text{ and } \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{2}$$

(c) Calculate one step with integration method (1), in which $\Delta t = 0.5$ and $t_0 = 0$, applied to (2) and use the given initial conditions. (2pt.)

(d) Show that the amplification factor for this integration method is given by: $Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2}$. (2pt.)

(e) Examine for which step sizes Δt, the integration method (1), applied to the initial value problem (2), is stable. (2pt.)

(f) This problem can also be solved in a decoupled way, that means one can use the method to solve first $x_2' = -4x_2 + \cos \pi t$ and after that to solve $x_1' = -2x_1 + x_2$ What is the advantage of this approach (+motivation)? (1pt.)

please turn over

2. In this exercise an estimate is determined for the acceleration of a vehicle. The measured distances of the vehicle from the starting line are given in the table below.

t (s)	0	10	20
$d(t)$ (m)	0	40	100

(a) We look for a difference formula of the second derivative of d in $2h$ of the form:

$$d''(2h) \approx Q(h) = \frac{\alpha_0}{h^2}d(0) + \frac{\alpha_1}{h^2}d(h) + \frac{\alpha_2}{h^2}d(2h).$$

In the remainder of this exercise we use this formula. Show that the coefficients $\alpha_0,\ \alpha_1$ and α_2 should satisfy the next system:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h^2} & + & \frac{\alpha_1}{h^2} & + & \frac{\alpha_2}{h^2} & = & 0\,, \\ -2\frac{\alpha_0}{h} & - & \frac{\alpha_1}{h} & & & = & 0\,, \\ 2\alpha_0 & + & \frac{1}{2}\alpha_1 & & & = & 1\,. \end{array}$$

(2 pt.)

(b) The solution of this system is given by $\alpha_0 = 1$, $\alpha_1 = -2$ and $\alpha_2 = 1$. Determine for these values an expression for the truncation error $d''(2h) - Q(h)$. (2 pt.)

(c) Give an estimate of the acceleration at $t = 20$. (1 pt.)

3. We want to find a root of the function $f(x) = -x^3 + 6x - \frac{23}{8}$.

(a) We choose to use the *fixed point iteration* $p_{n+1} = g(p_n)$, with $g(x) = \frac{x^3}{6} + \frac{23}{48}$ to find a root. *Show* that a fixed point of $g(x)$ is also a root of $f(x)$. (1 pt.)

(b) We start the fixed point iteration at $p_0 = 1$. *Calculate* p_1, p_2 and p_3 to four decimals and *sketch* the fixed point iteration in a figure. (2 pt.)

(c) Show that the chosen fixed point iteration *converges* for all $p_0 \in [0, 1]$. (2 pt.)

For the answers of this test we refer to:
https://diamhomes.ewi.tudelft.nl/~kvuik/wi3097/tentamen.html