DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400) Thursday June 27 2024, 13:30-16:30

Number of questions: This is an exam with 11 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. A method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$\begin{cases} k_1 &= \Delta t\, f\,(t_n, w_n) \\ k_2 &= \Delta t\, f\left(t_n + \frac{1}{2}\Delta t, w_n + \frac{1}{2}k_1\right) \\ k_3 &= \Delta t\, f\,(t_n + \Delta t, w_n - k_1 + 2k_2) \\ w_{n+1} &= w_n + (\alpha k_1 + \beta k_2 + \gamma k_3) \end{cases} \tag{1}$$

where Δt denotes the time–step and w_n represents the numerical solution at time t_n.

(a) The *amplification factor* of this method is given by

$$Q(\lambda\Delta t) = 1 + (\alpha + \beta + \gamma)\,\lambda\Delta t + \left(\frac{\beta}{2} + \gamma\right)(\lambda\Delta t)^2 + \gamma\,(\lambda\Delta t)^3\,.$$

Derive this amplification factor for the given method. ($2\frac{1}{2}$ pt.)

(b) Show that the *local truncation error* of the given method for the test equation $y' = \lambda y$ is $\mathcal{O}(\Delta t^3)$ *only* for $\alpha = \gamma = \frac{1}{6}$ and $\beta = \frac{2}{3}$. ($2\frac{1}{2}$ pt.)

(c) Given the initial value problem

$$\begin{cases} 2\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 2t, \\ \\ y(0) = 1, \quad \dfrac{dy}{dt}(0) = 1. \end{cases} \tag{2}$$

Show that this problem can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix}. \tag{3}$$

Give also the initial conditions for $x_1(0)$ and $x_2(0)$. ($1\frac{1}{2}$ pt.)

(d) Take $\alpha = \gamma = \frac{1}{6}$ and $\beta = \frac{2}{3}$.
Is the given method applied to this initial value problem stable for $\Delta t = 2$? ($1\frac{1}{2}$ pt.)

(e) Perform *one step* with the given method with $\Delta t = 2$, $t_0 = 0$, $\alpha = \gamma = \frac{1}{6}$ and $\beta = \frac{2}{3}$ for the initial value problem and the given initial conditions from (2). (2 pt.)

please turn over

2. We consider the one-dimensional convection–diffusion equation with Dirichlet boundary conditions:

$$\begin{cases} -u'' + u' = 1, & 0 \le x \le 1, \\ u(0) = 0, & u(1) = 0, \end{cases} \tag{4}$$

where $u = u(x)$, $u' = \frac{du}{dx}$ and $u'' = \frac{d^2u}{dx^2}$

(a) Show that

$$u(x) = x - \frac{1 - e^x}{1 - e} \tag{5}$$

is the exact solution to the boundary value problem (4). (1 pt.)

(b) We solve the boundary value problem (4) using finite differences, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the stepsize. Give a discretization method (+proof) where the truncation error is of order $O((\Delta x)^2)$. Take the boundary conditions into account. (2 pt.)

(c) Use a step size of $\Delta x = \frac{1}{4}$ to derive the system of equations $A\mathbf{y} = \mathbf{b}$. Take care of the boundary conditions. The system must have three unknowns and three equations, i.e. A is a 3×3 matrix and $\mathbf{y}$ and $\mathbf{b}$ are 3×1 column vectors. You do **not** have to solve this system. (2 pt.)

3. We want to find an approximation of the zero p of a function f, i.e. we want to find p such that $f(p) = 0$. However, we do not known the function f, but only some values of f in some points x are known, which are given in the table to the right.

x	$f(x)$
1	-1
$4/3$	$-2/9$
$7/5$	$-1/25$
$10/7$	$2/49$
2	2

Therefore we consider using the Secant method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{K_{n-1}}, \tag{6}$$

with K_{n-1} an approximation to $f'(p_{n-1})$, and in which p_{n-2}, p_{n-1} and p_n are three consecutive approximations of the zero p.
Equation (6) is based on formulating the linear interpolation polynomial L of f based on p_0 and p_1:

$$L(x) = f(p_0) + \frac{f(p_1) - f(p_0)}{p_1 - p_0}(x - p_0),$$

after which p_2 is found by solving $L(p_2) = 0$ for p_2.

(a) *Show* that, for $n = 2$, K_1 is given by

$$K_1 = \frac{f(p_1) - f(p_0)}{p_1 - p_0},$$

by solving $L(p_2) = 0$ for p_2. (2 pt.)

(b) Take $p_0 = 1$ and $p_1 = 2$. *Approximate* the zero p of f by calculating p_2. Hint: you may round p_2 to a value of x as given in the table. (1 pt.)

(c) Repeat the above steps with $n = 3$ by *stating* the formula for K_2 and *calculating* p_3. Hint: you may round p_3 to a value of x as given in the table. (2 pt.)

For the answers of this test we refer to:
https://diamhomes.ewi.tudelft.nl/~kvuik/wi3097/tentamen.html