DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400)

Tuesday July 16 2024, 13:30-16:30

1. (a) Consider the test equation $y' = \lambda y$, then it follows that

$$w_{n+1} = w_n + (1-\theta)\lambda\Delta t w_n + \theta\lambda\Delta t w_{n+1}.$$

Solving for w_{n+1} gives

$$w_{n+1} = \frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t} w_n.$$

Hence the amplification factor is given by

$$Q(\lambda\Delta t) = \frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t}.$$

(b) The local truncation error for the test equation $y' = \lambda y$ is given by

$$\tau_{n+1}(\Delta t) = \frac{e^{\lambda\Delta t} - Q(\lambda\Delta t)}{\Delta t} y_n.$$

The Taylor Series around 0 for $e^{\lambda\Delta t}$ is:

$$e^{\lambda\Delta t} = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3).$$

The Taylor Series around 0 for $Q(\lambda\Delta t)$ is:

$$\begin{aligned} Q(\lambda\Delta t) &= (1+(1-\theta)\lambda\Delta t)\frac{1}{1-\theta\lambda\Delta t} \\ &= (1+(1-\theta)\lambda\Delta t)\left(1+\theta\lambda\Delta t + \theta^2(\lambda\Delta t)^2 + \mathcal{O}\left(\Delta t^3\right)\right) \\ &= 1 + \lambda\Delta t + \theta(\lambda\Delta t)^2 + \mathcal{O}\left(\Delta t^3\right). \end{aligned}$$

Hence, this gives

$$e^{\lambda\Delta t} - Q(\lambda\Delta t) = \left(\frac{1}{2} - \theta\right)\left(\lambda\Delta t^2\right) + \mathcal{O}\left(\Delta t^3\right),$$

and hence

$$\begin{aligned} \tau_{n+1}(\Delta t) &= \frac{\left(\frac{1}{2}-\theta\right)\left(\lambda\Delta t^2\right) + \mathcal{O}\left(\Delta t^3\right)}{\Delta t} y_n \\ &= \left(\frac{1}{2}-\theta\right)(\lambda\Delta t)\, y_n + \mathcal{O}\left(\Delta t^2\right) = \mathcal{O}(\Delta t). \end{aligned}$$

Furthermore, $\tau_{n+1} = \mathcal{O}(\Delta t^2)$ if and only if $\theta = \frac{1}{2}$.

(c) To this extent, we determine the eigenvalues of the matrix. Subsequently, these eigenvalues are substituted into the amplification factor. The eigenvalues of the matrix are given by $-1 \pm 3i$.

Using $\Delta t = 1$, $\theta = \frac{1}{2}$ and taking $\lambda = -1 - 3i$ (alternatively, $\lambda = -1 + 3i$), it follows that

$$Q(\lambda \Delta t) = \frac{1 + \frac{1}{2}(-1 - 3i)}{1 - \frac{1}{2}(-1 - 3i)} = \frac{\frac{1}{2} - \frac{3}{2}i}{\frac{3}{2} + \frac{3}{2}i}.$$

Herewith, it follows that $|Q(\lambda \Delta t)|^2 = \frac{5}{9} \leq 1$. (Different methods to show this are possible.)

As the two eigenvalues are each others complex conjugate, only one eigenvalue has to be considered during the stability analysis. (Also correct: Repeating the above calculations for the other eigenvalue.)

Hence for $\Delta t = 1$ and $\theta = \frac{1}{2}$ it follows that the method applied to the given system is stable.

(d) The given method, applied to the system $\underline{x}' = A\underline{x}$ as given in the question and taking $\theta = \frac{1}{2}$, gives

$$\underline{w}_{n+1} = \underline{w}_n + \frac{1}{2}\Delta t A \underline{w}_n + \frac{1}{2}\Delta t A \underline{w}_{n+1}.$$

Rearranging gives the linear system

$$\left(I - \frac{1}{2}\Delta t A\right)\underline{w}_{n+1} = \left(I + \frac{1}{2}\Delta t A\right)\underline{w}_n.$$

With $\Delta t = 1$ and the initial condition, $\underline{w}_0 = \begin{bmatrix}1 & 0\end{bmatrix}^T$, this gives

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \underline{w}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.$$

Solving for $\underline{w}_1$ gives

$$\underline{w}_1 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

2. (a) $p = \sqrt{3}$ is a fixed point of the function g if $g(\sqrt{3}) = \sqrt{3}$. We calculate $g(p)$:

$$\begin{aligned} g(p) &= g(\sqrt{3}) \\ &= -\frac{1}{3}(\sqrt{3})^2 + \sqrt{3} + 1 \\ &= -1 + \sqrt{3} + 1 \\ &= \sqrt{3}, \end{aligned}$$

So $p = \sqrt{3}$ is indeed a fixed-point of the function g.

(b) The function g is a polynomial and polynomials are continuous everywhere, so there for g also is continuous on the interval $[1, 2]$.

(c) First note that g is a parabola opening to the bottom and therefore g has a maximum at the point where $g'(x) = 0$. We solve this equation:

$$\begin{aligned} & & g'(x) &= 0 \\ \Rightarrow & & -\frac{2}{3}x + 1 &= 0 \\ \Rightarrow & & x &= \frac{3}{2}, \end{aligned}$$

so the position of the maximum of g is located in the interval $[1, 2]$ and attains the value $g(3/2) = 7/4$. Therefore we conclude

$$g(x) \leq 2 \quad \text{for } x \in [1, 2].$$

The function g attains its minimum on the boundary of the interval $[1, 2]$, so evaluation of g at these points gives

$$\begin{aligned} g(1) &= 5/3, \\ g(2) &= 5/3. \end{aligned}$$

Therefore we conclude

$$g(x) \geq 1 \quad \text{for } x \in [1, 2].$$

Putting everything together, we have found

$$1 \leq g(x) \leq 2, \quad \text{for } x \in [1, 2].$$

as requested.

(d) The derivative of g is given by

$$g'(x) = -\frac{2}{3}x + 1,$$

which is a monotonous decreasing function. Therefore the minimum and maximum value are located on the boundary of the interval, leading to

$$\begin{aligned} & g'(2) &\leq \quad & g'(x) &\leq \quad & g'(1) \\ \Rightarrow \quad & -1/3 &\leq \quad & g'(x) &\leq \quad & 1/3 \\ \Rightarrow \quad & & & |g'(x)| &\leq \quad & 1/3 \end{aligned}$$

So $k = 1/3$.

(e) *Remark: The final value of* p_1 *should be given in 4 significant digits. Failure to do so results in a deduction of* $\frac{1}{4}$ *point if* $p_1 = 5/3$ *is stated.*
Remark: Calculation of p_2 *using* $p_1 = 5/3$ *is incorrect, and causes a deduction of* $\frac{1}{4}$ *point.*
Remark: The final value of p_2 *should be given in 4 significant digits. Failure to do so results in a deduction of* $\frac{1}{4}$ *point if* $p_2 = 47/27$ *is stated.*
Straightforward application of the fixed point iteration gives

$$\begin{aligned} p_1 &= g(p_0) \\ &= g(2.000) \\ &= 1.667, \end{aligned}$$

and

$$\begin{aligned} p_2 &= g(p_1) \\ &= g(1.667) \\ &= 1.741. \end{aligned}$$

3. (a) The **linear Lagrangian interpolatory polynomial**, with nodes x_0 and x_1, is given by

$$L_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1). \tag{1}$$

This is evident from application of the given formula.

(b) The **quadratic Lagrangian interpolatory polynomial** with nodes x_0, x_1 and x_2 is given by

$$\begin{aligned} L_2(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) && (2) \\ &+ \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) && (3) \\ &+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2). && (4) \end{aligned}$$

This is also evident from application of the given formula.

(c) Obviously, $L_1(3) = 6$ and $L_2(3) = 6$ since the Lagrange interpolation polynomial satisfies $L_n(x_k) = f(x_k)$ for all points $x_0, x_1, \ldots, x_n$. Next, we compute $L_1(2)$ and $L_2(2)$ for both linear and quadratic Lagrangian interpolation as approximations at $x = 2$. For **linear interpolation**, we have

$$L_1(2) = \frac{2 - 3}{1 - 3} \cdot 3 + \frac{2 - 1}{3 - 1} \cdot 6 = \frac{9}{2}, \tag{5}$$

and for **quadratic interpolation**, one obtains

$$L_2(2) = \frac{(2 - 3)(2 - 4)}{(1 - 3)(1 - 4)} \cdot 3 + \frac{(2 - 1)(2 - 4)}{(3 - 1)(3 - 4)} \cdot 6 + \frac{(2 - 1)(2 - 3)}{(4 - 1)(4 - 3)} \cdot 5 = \frac{16}{3}. \tag{6}$$