DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

Examiner responsible: C. Vuik
Examination reviewer: D. den Ouden-van der Horst

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(CTB2400)
Tuesday July 16 2024, 13:30-16:30

Number of questions: This is an exam with 12 open questions, subdivided in 3 main questions.
Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.
Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.
Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. A method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$w_{n+1} = w_n + (1-\theta)\Delta t f(t_n, w_n) + \theta \Delta t f(t_{n+1}, w_{n+1}),$$

where Δt denotes the time–step, w_n represents the numerical solution at time t_n and $0 \leq \theta \leq 1$.

(a) The *amplification factor* of this method is given by

$$Q(\lambda \Delta t) = \frac{1 + (1-\theta)\lambda \Delta t}{1 - \theta \lambda \Delta t}.$$

Derive this amplification factor for the given method. ($1\frac{1}{2}$ pt.)

(b) Show that the *local truncation error* of the given method is $\mathcal{O}(\Delta t)$ in general for the test equation $y' = \lambda y$. Also determine for which value of θ the method is $\mathcal{O}(\Delta t^2)$. ($3\frac{1}{2}$ pt.)
Hint: $e^x = 1 + x + \frac{1}{2}x^2 + \mathcal{O}(x^3)$.
Hint: $\frac{1}{1-x} = 1 + x + x^2 + \mathcal{O}(x^3)$ for $|x| < 1$.

(c) Take $\theta = \frac{1}{2}$. Given is the initial value problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{1}$$

with initial conditions $x_1(0) = 1, x_2(0) = 0$.
Is the given method applied to this initial value problem *stable* for $\Delta t = 1$? ($3\frac{1}{2}$ pt.)

(d) Perform *one step* with the given method with $\Delta t = 1$, $\theta = \frac{1}{2}$ and $t_0 = 0$ for the initial value problem (1) and the given initial conditions. ($1\frac{1}{2}$ pt.)

please turn over

2. We want to find an approximation of $\sqrt{3}$. Therefore we consider the fixed-point problem

$$x = g(x),$$

on the interval $[1, 2]$, where the function g is defined as

$$g(x) = -\frac{1}{3}x^2 + x + 1. \tag{2}$$

In the next exercises you will prove that $p = \sqrt{3}$ is indeed the unique fixed point of g in the interval $[1, 2]$ and that for any starting value of $p_0 \in [1, 2]$ the fixed-point iteration

$$p_{n+1} = g(p_n), \tag{3}$$

converges to $p = \sqrt{3}$, and you will perform this fixed-point iteration.

(a) *Show* that $p = \sqrt{3}$ is a fixed point of the function g. (1 pt.)

(b) *Argue* why g is continuous on $[1, 2]$. (1 pt.)

(c) *Show* that $1 \leq g(x) \leq 2$ for all $x \in [1, 2]$. (1 pt.)

(d) *Find* the smallest value k such that $|g'(x)| \leq k < 1$ for all $x \in [1, 2]$. (1 pt.)

(e) *Approximate* $p = \sqrt{3}$ by calculating p_1 and p_2 with 4 significant digits, given that $p_0 = 2.000$. (1 pt.)

3. We analyse **Lagrangian interpolation**. For given points $x_0,\ x_1,\ \ldots,\ x_n$, with their respective function values $f(x_0),\ f(x_1),\ \ldots,\ f(x_n)$, the interpolatory polynomial $L_n(x)$ is given by

$$L_n(x) = \sum_{k=0}^{n} f(x_k) L_{kn}(x), \text{ with}$$
$$L_{kn}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}. \tag{4}$$

(a) Give the *linear Lagrangian interpolatory polynomial* $L_1(x)$ with nodes x_0 and x_1. (1 pt.)

(b) Give the *quadratic Lagrangian interpolatory polynomial* $L_2(x)$ with nodes x_0, x_1 and x_2. (2 pt.)

(c) Calculate $L_n(2)$ and $L_n(3)$ both by using linear and quadratic Lagrangian interpolation using the following measured values:

k	x_k	$f(x_k)$
0	1	3
1	3	6
2	4	5

(2 pt.)

For the answers of this test we refer to:
https://diamhomes.ewi.tudelft.nl/~kvuik/wi3097/tentamen.html