DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(CTB2400)
Thursday June 26 2025, 13:30-16:30

1. (a) The amplification factor is defined by

$$Q(\lambda\Delta t) = \frac{w_{n+1}}{w_n},$$

where w_{n+1} results from applying one step of the method to the test equation $y' = \lambda y$. First we calculate k_1 and use $f(t,y) = \lambda y$:

$$\begin{aligned} & & k_1 &= \lambda\left(w_n + \theta\Delta t k_1\right) \\ &\Rightarrow & k_1 &= \lambda w_n + \theta\lambda\Delta t k_1 \\ &\Rightarrow & k_1 - \theta\lambda\Delta t k_1 &= \lambda w_n \\ &\Rightarrow & \left(1-\theta\lambda\Delta t\right)k_1 &= \lambda w_n \\ &\Rightarrow & k_1 &= \frac{\lambda}{1-\theta\lambda\Delta t}w_n \end{aligned}$$

Then we calculate w_{n+1}:

$$\begin{aligned} w_{n+1} &= w_n + \Delta t k_1 \\ &= w_n + \frac{\lambda\Delta t}{1-\theta\lambda\Delta t}w_n \\ &= \left(1 + \frac{\lambda\Delta t}{1-\theta\lambda\Delta t}\right)w_n \\ &= \frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t}w_n \end{aligned}$$

Finally division by w_n gives

$$Q(\lambda\Delta t) = \frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t}.$$

(b) The local truncation error for the test equation is given by

$$\tau_{n+1}(\Delta t) = \frac{e^{\lambda\Delta t} - Q(\lambda\Delta t)}{\Delta t}y_n., \tag{1}$$

$e^{\lambda\Delta t}$ can be expanded by the use of Taylor expansions around $\Delta t = 0$:

$$e^{\lambda\Delta t} = 1 + \lambda\Delta t + \frac{1}{2}\left(\lambda\Delta t\right)^2 + \mathcal{O}(\Delta t^3).$$

$\frac{1}{1-\frac{1}{2}\lambda\Delta t}$ can be expanded by the use of Taylor expansions around $\Delta t = 0$:

$$\begin{aligned} \frac{1}{1-\frac{1}{2}\lambda\Delta t} &= 1 + \frac{1}{2}\lambda\Delta t + \left(\frac{1}{2}\lambda\Delta t\right)^2 + \mathcal{O}(\Delta t^3) \\ &= 1 + \frac{1}{2}\lambda\Delta t + \frac{1}{4}\left(\lambda\Delta t\right)^2 + \mathcal{O}(\Delta t^3). \end{aligned}$$

This means the amplification factor can be rewritten to:

$$\begin{aligned}
Q(\lambda\Delta t) &= \left(1+\frac{1}{2}\lambda\Delta t\right)\frac{1}{1-\frac{1}{2}\lambda\Delta t} \\
&= \left(1+\frac{1}{2}\lambda\Delta t\right)\left(1+\frac{1}{2}\lambda\Delta t+\frac{1}{4}\left(\lambda\Delta t\right)^2+\mathcal{O}(\Delta t^3)\right) \\
&= 1+\lambda\Delta t+\frac{1}{2}\left(\lambda\Delta t\right)^2+\mathcal{O}(\Delta t^3).
\end{aligned}$$

Substitution of the above in the local truncation error results in:

$$\begin{aligned}
\tau_{n+1} &= \frac{e^{\lambda\Delta t}-Q(\lambda\Delta t)}{\Delta t}y_n \\
&= \frac{\left(1+\lambda\Delta t+\frac{1}{2}\left(\lambda\Delta t\right)^2+\mathcal{O}(\Delta t^3)\right)-\left(1+\lambda\Delta t+\frac{1}{2}\left(\lambda\Delta t\right)^2+\mathcal{O}(\Delta t^3)\right)}{\Delta t}y_n \\
&= \frac{\mathcal{O}(\Delta t^3)}{\Delta t}y_n \\
&= \mathcal{O}(\Delta t^2).
\end{aligned}$$

(c) For stability,

$$|Q(\lambda\Delta t)| \le 1,$$

must hold for all eigenvalues of the linear initial value problem, with Q the amplification factor of the given method.
First, we determine the eigenvalues of the matrix A. Subsequently, the eigenvalues are substituted into the amplification factor.
The eigenvalues of the matrix A are given by $\lambda_1 = -3$ and $\lambda_2 = -1$.
We first consider $\lambda_1 = -3$:

$$\begin{aligned}
Q(\lambda_1\Delta t) &= \frac{1-\frac{3}{2}\Delta t}{1+\frac{3}{2}\Delta t} \\
&= \frac{2-3\Delta t}{2+3\Delta t}.
\end{aligned}$$

Applying the stability criteria results in

$$-1 \le \frac{2-3\Delta t}{2+3\Delta t} \le 1,$$

and multiplying with the denominator gives

$$-2-3\Delta t \le 2-3\Delta t \le 2+3\Delta t.$$

First we solve the left inequality:

$$\begin{aligned}
& -2-3\Delta t \le 2-3\Delta t \\
\Rightarrow\quad & -2 \le 2
\end{aligned}$$

As this inequality is always true, we obtain no new information.
Then we solve the right inequality:

$$\begin{aligned}
& 2-3\Delta t \le 2+3\Delta t \\
\Rightarrow\quad & 2-6\Delta t \le 2 \\
\Rightarrow\quad & -6\Delta t \le 0
\end{aligned}$$

As this inequality is always true for $\Delta t > 0$, we obtain no new information.
Repeating this for $\lambda_2 = -1$ also results in no new information.
Therefor the time integration method applied to the initial value problem is stable for

$$\Delta t > 0.$$

(d) First we calculate $\mathbf{k}_1$, where we use $\Delta t = 1$:

$$\begin{aligned}
& & \mathbf{k}_1 &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\mathbf{k}_1 \right) + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Rightarrow & & \mathbf{k}_1 &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{k}_1 \\
\Rightarrow & & \mathbf{k}_1 - \frac{1}{2} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{k}_1 &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow & & \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix} \mathbf{k}_1 &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\
\Rightarrow & & \mathbf{k}_1 &= \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\
\Rightarrow & & \mathbf{k}_1 &= \begin{pmatrix} -0.9333 \\ 0.2667 \end{pmatrix}.
\end{aligned}$$

Then we calculate $\mathbf{w}_1$, again with $\Delta t = 1$:

$$\begin{aligned}
\mathbf{w}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -0.9333 \\ 0.2667 \end{pmatrix} \\
&= \begin{pmatrix} -0.0667 \\ 0.2667 \end{pmatrix}.
\end{aligned}$$

2. (a) $p = \sqrt{3}$ is a fixed point of the function g if $g(\sqrt{3}) = \sqrt{3}$. We calculate $g(p)$:

$$\begin{aligned} g(p) &= g(\sqrt{3}) \\ &= -\frac{1}{3}(\sqrt{3})^2 + \sqrt{3} + 1 \\ &= -1 + \sqrt{3} + 1 \\ &= \sqrt{3}, \end{aligned}$$

So $p = \sqrt{3}$ is indeed a fixed-point of the function g.

(b) The function g is a polynomial and polynomials are continuous everywhere, so there for g also is continuous on the interval $[1, 2]$.

(c) First note that g is a parabola opening to the bottom and therefore g has a maximum at the point where $g'(x) = 0$. We solve this equation:

$$\begin{aligned} & & g'(x) &= 0 \\ &\Rightarrow & -\frac{2}{3}x + 1 &= 0 \\ &\Rightarrow & x &= \frac{3}{2}, \end{aligned}$$

so the position of the maximum of g is located in the interval $[1, 2]$ and attains the value $g(3/2) = 7/4$. Therefore we conclude

$$g(x) \leq 2 \quad \text{for } x \in [1, 2].$$

The function g attains its minimum on the boundary of the interval $[1, 2]$, so evaluation of g at these points gives

$$\begin{aligned} g(1) &= 5/3, \\ g(2) &= 5/3. \end{aligned}$$

Therefore we conclude

$$g(x) \geq 1 \quad \text{for } x \in [1, 2].$$

Putting everything together, we have found

$$1 \leq g(x) \leq 2, \quad \text{for } x \in [1, 2].$$

as requested.

(d) The derivative of g is given by

$$g'(x) = -\frac{2}{3}x + 1,$$

which is a monotonous decreasing function. Therefore the minimum and maximum value are located on the boundary of the interval, leading to

$$\begin{array}{rcccc} & g'(2) & \leq & g'(x) & \leq & g'(1) \\ \Rightarrow & -1/3 & \leq & g'(x) & \leq & 1/3 \\ \Rightarrow & & & |g'(x)| & \leq & 1/3 \end{array}$$

So $k = 1/3$.

(e) *Remark: The final value of p_1 should be given in 4 significant digits. Failure to do so results in a deduction of $\frac{1}{4}$ point if $p_1 = 5/3$ is stated.*
Remark: Calculation of p_2 using $p_1 = 5/3$ is incorrect, and causes a deduction of $\frac{1}{4}$ point.
Remark: The final value of p_2 should be given in 4 significant digits. Failure to do so results in a deduction of $\frac{1}{4}$ point if $p_2 = 47/27$ is stated.
Straightforward application of the fixed point iteration gives

$$\begin{aligned} p_1 &= g(p_0) \\ &= g(2.000) \\ &= 1.667, \end{aligned}$$

and

$$\begin{aligned} p_2 &= g(p_1) \\ &= g(1.667) \\ &= 1.741. \end{aligned}$$

3. (a) Using central differences for the second order derivative at a node $x_j = j\Delta x$ gives

$$y''(x_j) \approx \frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta x^2} =: Q(\Delta x). \tag{2}$$

Here, $y_j := y(x_j)$. Next, we will prove that this approximation is second order accurate, that is $|y''(x_j) - Q(\Delta x)| = \mathcal{O}(\Delta x^2)$.
Using Taylor's Theorem around $x = x_j$ gives

$$\begin{aligned} y_{j+1} &= y(x_j + \Delta x) = y(x_j) + \Delta x y'(x_j) + \tfrac{\Delta x^2}{2} y''(x_j) + \tfrac{\Delta x^3}{3!} y'''(x_j) + \tfrac{\Delta x^4}{4!} y''''(\eta_+), \\ y_{j-1} &= y(x_j - \Delta x) = y(x_j) - \Delta x y'(x_j) + \tfrac{\Delta x^2}{2} y''(x_j) - \tfrac{\Delta x^3}{3!} y'''(x_j) + \tfrac{\Delta x^4}{4!} y''''(\eta_-). \end{aligned} \tag{3}$$

Here, η_+ and η_- are numbers within the intervals (x_j, x_{j+1}) and (x_{j-1}, x_j), respectively. Substitution of these expressions into $Q(\Delta x)$ gives

$$|y''(x_j) - Q(\Delta x)| = \mathcal{O}(\Delta x^2).$$

This leads to the following discretisation formula for internal grid nodes:

$$\frac{-w_{j-1} + 2w_j - w_{j+1}}{\Delta x^2} + (x_j + 1)w_j = 1. \tag{4}$$

Here, w_j represents the numerical approximation of the solution y_j. To deal with the boundary $x = 0$, we use a virtual node at $x = -\Delta x$, and we define $y_{-1} := y(-\Delta x)$. Then, using central differences at $x = 0$ gives

$$0 = y'(0) \approx \frac{y_1 - y_{-1}}{2\Delta x} =: Q_b(\Delta x). \tag{5}$$

Using Taylor's Theorem, gives

$$\begin{aligned} Q_b(\Delta x) &= \\ &= \frac{y(0) + \Delta x y'(0) + \frac{\Delta x^2}{2} y''(0) + \frac{\Delta x^3}{3!} y'''(\eta_+)}{2\Delta x} \\ &- \frac{y(0) - \Delta x y'(0) + \frac{\Delta x^2}{2} y''(0) - \frac{\Delta x^3}{3!} y'''(\eta_-)}{2\Delta x} \\ &= y'(0) + \mathcal{O}(\Delta x^2). \end{aligned}$$

Again, we get an error of $\mathcal{O}(\Delta x^2)$.

(b) With respect to the numerical approximation at the virtual node, we get

$$\frac{w_1 - w_{-1}}{2\Delta x} = 0 \quad \Leftrightarrow \quad w_{-1} = w_1. \tag{6}$$

The discretisation at $x = 0$ is given by

$$\frac{-w_{-1} + 2w_0 - w_1}{\Delta x^2} + w_0 = 1. \tag{7}$$

Substitution of equation (6) into the above equation, yields

$$\frac{2w_0 - 2w_1}{\Delta x^2} + w_0 = 1. \tag{8}$$

Subsequently, we consider the boundary $x = 1$. To this extent, we consider its neighbouring point x_{n-1} and substitute the boundary condition $w_n = y(1) = y_n = 1$ into equation (4) to obtain

$$\begin{aligned} & \frac{-w_{n-2} + 2w_{n-1}}{\Delta x^2} + (x_{n-1} + 1)w_{n-1} && (9) \\ = \; & 1 && (10) \\ = \; & 1 + \frac{1}{\Delta x^2}. && (11) \end{aligned}$$

This concludes our discretisation of the boundary conditions. In order to get a symmetric discretisation matrix, one divides equation (8) by 2.

Next, we use $\Delta x = 1/3$. From equations (4, 8, 11) we obtain the following system

$$\begin{aligned} 9\frac{1}{2}w_0 - 9w_1 &= \frac{1}{2} \\ -9w_0 + 19\frac{1}{3}w_1 - 9w_2 &= 1 \\ -9w_1 + 19\frac{2}{3}w_2 &= 10. \end{aligned}$$

(c) The Gershgorin circle theorem states that the eigenvalues of a square matrix $\mathbf{A}$ are located in the complex plane in the union of circles

$$|z - a_{ii}| \leq \sum_{\substack{j\neq i \\ j=1}}^{n} |a_{ij}| \quad \text{where} \quad z \in \mathbb{C} \tag{12}$$

For the 3×3 matrix derived in part (b) we have

- For $i = 1$:
$$\left|z - 9\frac{1}{2}\right| \leq 9 \quad \Rightarrow \quad |\lambda_1|_{\min} \geq \frac{1}{2} \tag{13}$$
- For $i = 2$:
$$\left|z - 19\frac{1}{3}\right| \leq 18 \quad \Rightarrow \quad |\lambda_2|_{\min} \geq 1\frac{1}{3} \tag{14}$$
- For $i = 3$:
$$\left|z - 19\frac{2}{3}\right| \leq 9 \quad \Rightarrow \quad |\lambda_3|_{\min} \geq 10\frac{2}{3} \tag{15}$$

Hence, a lower bound for the smallest eigenvalue is $\frac{1}{2}$. For a symmetric matrix $\mathbf{A}$ we have

$$\|\mathbf{A}^{-1}\| = \frac{1}{|\lambda|_{\min}} \leq 2 \tag{16}$$

This proves that the finite-difference scheme is stable, e.g., with constant $C = 2$.