DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

Examiner responsible: C. Vuik
Examination reviewer: D. den Ouden-van der Horst

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400) Thursday June 26 2025, 13:30-16:30

Number of questions: This is an exam with 12 open questions, subdivided in 3 main questions.
Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.
Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.
Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. A method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by:

$$\begin{cases} k_1 &= f(t_n + \theta\Delta t, w_n + \theta\Delta t k_1) \\ w_{n+1} &= w_n + \Delta t k_1 \end{cases} \tag{1}$$

where Δt is the time step, $\theta \in [0, 1]$ and w_n is the numerical approximation at time t_n.

(a) Give the amplification factor of this method. ($2\frac{1}{2}$ pt.)

In the remainder of this exercise we take $\theta = \frac{1}{2}$.

(b) For $\theta = \frac{1}{2}$ the amplification factor is given bij $Q(\lambda\Delta t) = \frac{1+\frac{1}{2}\lambda\Delta t}{1-\frac{1}{2}\lambda\Delta t}$. Show that the local truncation error of the given method is of the order $O(\Delta t^2)$ for the test equation $y' = \lambda y$.
(Hint: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$ and $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$) ($2\frac{1}{2}$ pt.)

Consider the following system of differential equations:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos(\pi t) \\ 0 \end{bmatrix}. \tag{2}$$

with intial conditions $x_1(0) = 1$ and $x_2(0) = 0$.

(c) Determine for which $\Delta t > 0$ the given method is stable when it is applied to the system as given above. ($2\frac{1}{2}$ pt.)

(d) Calculate one step with this method for $\Delta t = 1$ and $t_0 = 0$. ($2\frac{1}{2}$ pt.)

please turn over

2. We want to find an approximation of $\sqrt{3}$. Therefore we consider the fixed-point problem

$$x = g(x),$$

on the interval $[1, 2]$, where the function g is defined as

$$g(x) = -\frac{1}{3}x^2 + x + 1. \tag{3}$$

In the next exercises you will prove that $p = \sqrt{3}$ is indeed the unique fixed point of g in the interval $[1, 2]$ and that for any starting value of $p_0 \in [1, 2]$ the fixed-point iteration

$$p_{n+1} = g(p_n), \tag{4}$$

converges to $p = \sqrt{3}$, and you will perform this fixed-point iteration.

(a) *Show* that $p = \sqrt{3}$ is a fixed point of the function g. (1 pt.)

(b) *Argue* why g is continuous on $[1, 2]$. (1 pt.)

(c) *Show* that $1 \leq g(x) \leq 2$ for all $x \in [1, 2]$. (1 pt.)

(d) *Find* the smallest value k such that $|g'(x)| \leq k < 1$ for all $x \in [1, 2]$. (1 pt.)

(e) *Approximate* $p = \sqrt{3}$ by calculating p_1 and p_2 with 4 significant digits, given that $p_0 = 2.000$. (1 pt.)

3. We consider the boundary-value problem

$$\begin{cases} -y''(x) + (x+1)y(x) = 1, & 0 < x < 1, \\ y'(0) = 0, \quad y(1) = 1. \end{cases} \tag{5}$$

(a) We aim at solving the boundary value problem (5) using finite differences, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size.
Give a discretisation method (+proof) where
- the truncation error is of order $\mathcal{O}((\Delta x)^2)$;
- the boundary conditions are taken into account;
- and the discretisation matrix is symmetric.

Use a virtual point for the boundary condition at $x = 0$. Hint: in an interior point the equation is given by: $\frac{-w_{j-1}+2w_j-w_{j+1}}{\Delta x^2} + (x_j + 1)w_j = 1$. (2.5 pt.)

(b) Give the linear system of equations $\mathbf{A}\mathbf{w} = \mathbf{f}$ that results from applying the finite-difference scheme from (a) with three (after processing the virtual points) unknowns (i.e. $\Delta x = 1/3$). $\mathbf{A}$ is a 3×3 matrix. (1 pt.)

(c) Since the 3×3 system matrix $\mathbf{A}$ from (b) is symmetric, all eigenvalues are real.
Use the Gershgorin circle theorem to estimate the smallest eigenvalue $|\lambda|_{\min}$.
From that conclude that the finite-difference scheme from (a) is stable, that is, $\mathbf{A}^{-1}$ exists and there is a constant C such that $\|\mathbf{A}^{-1}\| \leq C$ for $\Delta x \to 0$. Hint: Gershgorin implies that all eigenvalues are contained in circles $|z - a_{ii}| \leq \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}|$ where $z \in \mathbb{C}$. (1.5 pt.)

For the answers of this test we refer to:
https://diamhomes.ewi.tudelft.nl/~kvuik/wi3097/tentamen.html