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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(CTB2400)
Tuesday July 15 2025, 13:30-16:30

1. (a) The local truncation error is given by

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}, \tag{1}$$

in which we determine y_{n+1} by the use of Taylor expansions around t_n:

$$y_{n+1} = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \mathcal{O}(\Delta t^3). \tag{2}$$

We bear in mind that

$$y'(t_n) = f(t_n, y_n)$$

$$\begin{aligned} y''(t_n) &= \frac{df(t_n,y_n)}{dt} = \frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y} y'(t_n) \\ &= \frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y} f(t_n,y_n). \end{aligned}$$

Hence

$$y_{n+1} = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2}\left(\frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y} f(t_n,y_n)\right) + \mathcal{O}(\Delta t^3). \tag{3}$$

After substitution of the predictor $z^*_{n+1} = y_n + \Delta t f(t_n, y_n)$ into the corrector, and after using a Taylor expansion around (t_n, y_n), we obtain for z_{n+1}:

$$\begin{aligned} z_{n+1} &= y_n + \frac{\Delta t}{2}\left(f(t_n,y_n) + f(t_n+\Delta t, y_n + \Delta t f(t_n,y_n))\right) \\ &= y_n + \frac{\Delta t}{2}\left(2f(t_n,y_n) + \Delta t\left(\frac{\partial f(t_n,y_n)}{\partial t} + f(t_n,y_n)\frac{\partial f(t_n,y_n)}{\partial y}\right) + \mathcal{O}(\Delta t^2)\right). \end{aligned}$$

Herewith, one obtains

$$y_{n+1} - z_{n+1} = \mathcal{O}(\Delta t^3), \text{ and hence } \tau_{n+1}(\Delta t) = \frac{\mathcal{O}(\Delta t^3)}{\Delta t} = \mathcal{O}(\Delta t^2). \tag{4}$$

(b) Let $x_1 = y$ and $x_2 = y'$, then $y'' = x_2'$, and hence

$$\begin{aligned} x_2' + \tfrac{4}{3}x_1 + 2x_2 &= \cos(t), \\ x_1' &= x_2. \end{aligned} \tag{5}$$

We write this as

$$\begin{cases} x_1' &= x_2, \\ x_2' &= -\tfrac{4}{3}x_1 - 2x_2 + \cos(t). \end{cases} \tag{6}$$

Finally, this is represented in the following matrix-vector form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{4}{3} & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}. \tag{7}$$

In which, we have the following matrix $A = \begin{pmatrix} 0 & 1 \\ -\frac{4}{3} & -2 \end{pmatrix}$ and $\underline{f} = \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}$. The initial conditions are defined by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) *Note: Every miscalculation in the calculation of* $\underline{w}_1^*$ *gives a subtraction of* 1/4 *point, with at most* 1/2 *point being subtracted.*

Note: The calculation of $\underline{w}_1$ *must be consistent with the value for* $\underline{w}_1^*$. *If not,* 1 *point is subtracted.*

Note: Every miscalculation in the calculation of $\underline{w}_1$ *gives a subtraction of* 1/4 *point, with at most* 1 *point being subtracted.*

Application of the integration method to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{aligned} \underline{w}_1^* &= \underline{w}_0 + \Delta t \left(A\underline{w}_0 + \underline{f}_0 \right), \\ \underline{w}_1 &= \underline{w}_0 + \tfrac{\Delta t}{2} \left(A\underline{w}_0 + f_0 + A\underline{w}_1^* + \underline{f}_1 \right). \end{aligned} \tag{8}$$

With the initial condition $\underline{w}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Delta t = 0.1$, this gives the following result for the predictor

$$\underline{w}_1^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{10} \left(\begin{pmatrix} 0 & 1 \\ -\frac{4}{3} & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -\frac{1}{30} \end{pmatrix} = \begin{pmatrix} 1 \\ -0.033 \end{pmatrix}. \tag{9}$$

The corrector is calculated as follows

$$\begin{aligned} \underline{w}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{20} \left(\begin{pmatrix} 0 & 1 \\ -\frac{4}{3} & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\frac{4}{3} & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{30} \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(\frac{1}{10}) \end{pmatrix} \right) \\ &= \begin{pmatrix} \frac{599}{600} \\ -\frac{121}{4000} \end{pmatrix} = \begin{pmatrix} 0.998 \\ -0.030 \end{pmatrix} \end{aligned}$$

(d) Consider the test equation $y' = \lambda y$, then one gets

$$\begin{aligned} w_{n+1}^* &= w_n + \Delta t \lambda w_n = (1 + \Delta t \lambda) w_n, \\ w_{n+1} &= w_n + \frac{\Delta t}{2} (\lambda w_n + \lambda w_{n+1}^*) \\ &= w_n + \frac{\Delta t}{2} (\lambda w_n + \lambda (w_n + \Delta t \lambda w_n)) \\ &= \left(1 + \Delta t \lambda + \frac{(\Delta t \lambda)^2}{2} \right) w_n. \end{aligned}$$

Hence the amplification factor is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2}. \tag{10}$$

(e) First, we determine the eigenvalues of the matrix A. Subsequently, the eigenvalues are substituted into the amplification factor.

The eigenvalues of the matrix A are given by $\lambda_1 = -1 + 0.5774i$ and $\lambda_2 = -1 - 0.5774i$.

The method is stable if $|Q(\lambda\Delta t)| \leq 1$ for both eigenvalues. Since the eigenvalues are complex valued, it is sufficient to check this condition only for λ_1.

Substituting λ_1 into the amplification factor leads to:

$$Q(\lambda_1\Delta t) = 1 + (-1 + 0.5774i)\Delta t + (0.3333 - 0.5774i)(\Delta t)^2$$

Note that $|Q(\lambda_1\Delta t)|^2 \leq 1$ implies $|Q(\lambda_1\Delta t)| \leq 1$. The first inequality leads to the condition:

$$(1 - \Delta t + 0.3333(\Delta t)^2)^2 + (0.5774\Delta t - 0.5774(\Delta t)^2)^2 \leq 1.$$

This is sufficient to obtain all points for this question. To find an explicit upperbound for Δt is not required.

2. (a) Because $s(x)$ consists of polynomials, the only possible point of discontinuity is the node $x = 0$, so $s(x)$ is continuous if it is continuous in $x = 0$.
Therefore we have to show

$$\lim_{x \to 0^-} s(x) = \lim_{x \to 0^+} s(x).$$

The left limit equals:

$$\begin{aligned} \lim_{x \to 0^-} s(x) &= \lim_{x \to 0^-} -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 \\ &= 2. \end{aligned}$$

The right limit equals:

$$\begin{aligned} \lim_{x \to 0^+} s(x) &= \lim_{x \to 0^+} \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 \\ &= 2. \end{aligned}$$

So $s(x)$ is continuous.
The derivative $s'(x)$ is given by

$$s'(x) = \begin{cases} -\frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} & \text{if} \quad x \in [-1, 0), \\ \frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} & \text{if} \quad x \in [0, 1]. \end{cases}$$

$s'(x)$ is continuous if it is continuous in $x = 0$, so we have to show

$$\lim_{x \to 0^-} s'(x) = \lim_{x \to 0^+} s'(x).$$

The left limit equals:

$$\begin{aligned} \lim_{x \to 0^-} s'(x) &= \lim_{x \to 0^-} -\frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

The right limit equals:

$$\begin{aligned} \lim_{x \to 0^+} s'(x) &= \lim_{x \to 0^+} \frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

So $s'(x)$ is continuous.
The second derivative $s''(x)$ is given by

$$s''(x) = \begin{cases} -\frac{9}{2}x - \frac{9}{2} & \text{if} \quad x \in [-1, 0), \\ \frac{9}{2}x - \frac{9}{2} & \text{if} \quad x \in [0, 1]. \end{cases}$$

$s''(x)$ is continuous if it is continuous in $x = 0$, so we have to show

$$\lim_{x \to 0^-} s''(x) = \lim_{x \to 0^+} s''(x).$$

The left limit equals:

$$\begin{aligned}\lim_{x\to 0^-} s''(x) &= \lim_{x\to 0^-} -\frac{9}{2}x - \frac{9}{2} \\ &= -\frac{9}{2}.\end{aligned}$$

The right limit equals:

$$\begin{aligned}\lim_{x\to 0^+} s''(x) &= \lim_{x\to 0^+} \frac{9}{2}x - \frac{9}{2} \\ &= -\frac{9}{2}.\end{aligned}$$

So $s''(x)$ is continuous.

(b) Evaluating $s''(x)$ in $x = -1$ gives:

$$s''(-1) = -\frac{9}{2}x - \frac{9}{2}\bigg|_{x=-1} = \frac{9}{2} - \frac{9}{2} = 0,$$

and evaluation at $x = 1$ gives

$$s''(1) = \frac{9}{2}x - \frac{9}{2}\bigg|_{x=1} = \frac{9}{2} - \frac{9}{2} = 0,$$

so indeed $s''(x) = 0$ in the end points.

(c) The nodes of the spline are $x = -1$, $x = 0$ and $x = 1$.
We will evaluate $s(x)$ in these three nodes and show that it is equal to $f(x)$ in these nodes:

$$\begin{aligned}s(-1) &= -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=-1} \\ &= -\frac{3}{4}(-1)^3 - \frac{9}{4}(-1)^2 + \frac{1}{2}(-1) + 2 \\ &= \frac{3}{4} - \frac{9}{4} - \frac{1}{2} + 2 \\ &= 0 \\ &= f(-1), \\ s(0) &= \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=0} \\ &= \frac{3}{4}(0)^3 - \frac{9}{4}(0)^2 + \frac{1}{2}(0) + 2 \\ &= 0 - 0 + 0 + 2 \\ &= 2 \\ &= f(0), \\ s(1) &= \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=1} \\ &= \frac{3}{4}(1)^3 - \frac{9}{4}(1)^2 + \frac{1}{2}(1) + 2 \\ &= \frac{3}{4} - \frac{9}{4} + \frac{1}{2} + 2 \\ &= 1 \\ &= f(1).\end{aligned}$$

(d) $x = -\frac{1}{2}$ lies in the left interval, so we need to perform the next calculation:

$$
\begin{aligned}
f\left(-\frac{1}{2}\right) &\approx s\left(-\frac{1}{2}\right) \\
&= -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=-\frac{1}{2}} \\
&= -\frac{3}{4}\left(-\frac{1}{2}\right)^3 - \frac{9}{4}\left(-\frac{1}{2}\right)^2 + \frac{1}{2}\left(-\frac{1}{2}\right) + 2 \\
&= \frac{41}{32} = 1.2812.
\end{aligned}
$$

3. (a) Newton–Raphson's Method is an iterative method to find $p \in \mathbb{R}$ such that $f(p) = 0$. One constructs a sequence of successive approximations $\{p_n\}$. Given the n–th estimate, then p_{n+1} is obtained through linearizing around p_n and by finding p_{n+1} by determining the point where the linearization (tangent) equals zero. Linearization of $f(p)$ around p_n gives (upon neglecting the error)

$$f(p) \approx f(p_n) + f'(p_n)(p - p_n) =: L(p; p_n), \tag{11}$$

for any p provided the second derivative of $f(p)$ is bounded and where $L(p; p_n)$ denotes the tangent (linearization) of $f(p)$ at point $(p_n, f(p_n))$. Then the next point is found upon setting $L(p_{n+1}; p_n) = 0$:

$$f(p_n) + f'(p_n)(p_{n+1} - p_n) = 0. \tag{12}$$

The above equation is solved for p_{n+1}, and gives

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \tag{13}$$

which is the famous Newton–Raphson formula for root–finding. For the graphical derivation, see Figure 4.2 in the book.

(b) The Jacobian matrix of $\mathbf{f}(\mathbf{x})$ is defined by

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_1}{\partial x_m}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_m}{\partial x_m}(\mathbf{x}) \end{pmatrix}.$$

The definition of the Newton–Raphson method is:

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n-1)} - \mathbf{J}^{-1}(\mathbf{p}^{(n-1)})\mathbf{f}(\mathbf{p}^{(n-1)}). \tag{14}$$

(c) First, we rewrite the system into the form

$$\begin{aligned} f_1(p_1, p_2) &= 0, \\ f_2(p_1, p_2) &= 0, \end{aligned} \tag{15}$$

by setting

$$\begin{aligned} f_1(p_1, p_2) &:= 2p_1 - p_2 + p_1 p_2, \\ f_2(p_1, p_2) &:= -1p_1 + 2p_2 + (p_2)^3 - 1. \end{aligned} \tag{16}$$

We denote the Jacobian matrix by $\mathbf{J}(p_1, p_2)$. Note that

$$\mathbf{J}(\mathbf{p}) = \begin{pmatrix} 2 + p_2^{(0)} & -1 + p_1^{(0)} \\ -1 & 2 + 3(p_2^{(0)})^2 \end{pmatrix}. \tag{17}$$

Using $p_1^{(0)} = p_2^{(0)} = 0$ we obtain:

$$\mathbf{J}(\mathbf{p}^{(0)}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{18}$$

This implies that

$$\mathbf{J}(\mathbf{p}^{(0)})^{-1} = \frac{1}{2^2 - 1}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{19}$$

Furthermore

$$\mathbf{f}(\mathbf{p}^{(0)}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \tag{20}$$

so

$$\mathbf{p}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{2^2 - 1}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}. \tag{21}$$