DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400)

Tuesday July 15 2025, 13:30-16:30

Number of questions: This is an exam with 12 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. For the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, we use the following integration method:

$$\begin{cases} w^*_{n+1} = w_n + \Delta t f(t_n, w_n) \\ w_{n+1} = w_n + \dfrac{\Delta t}{2}\left(f(t_n, w_n) + f(t_{n+1}, w^*_{n+1})\right). \end{cases} \tag{1}$$

Here Δt denotes the timestep and w_n represents the numerical approximation at time t_n.

(a) Show that the local truncation error of the integration method is of the order $\mathcal{O}(\Delta t^2)$. (You are not allowed to use the test equation here.) (3pt.)

Consider the following initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + \dfrac{4}{3}y = \cos t, \\ y(0) = 1, \quad \dfrac{dy}{dt}(0) = 0. \end{cases} \tag{2}$$

(b) Show that the above initial value problem can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{4}{3} & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}. \tag{3}$$

Give the initial conditions for $x_1(0)$ and $x_2(0)$ as well. (1pt.)

(c) Calculate one step with integration method (1), in which $\Delta t = 0.1$ and $t_0 = 0$, applied to (3) and use the given initial conditions. (2pt.)

(d) Derive the amplification factor for the integration method. (2pt.)

(e) Examine for which stepsizes Δt, the integration method (1), applied to the initial value problem (3), is stable. (2pt.)

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2. We have approximated a function f satisfying $f(-1) = 0$, $f(0) = 2$ and $f(1) = 1$ with a natural cubic spline s given by

$$s(x) = \begin{cases} -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 & \text{if} \quad x \in [-1, 0), \\ \\ \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 & \text{if} \quad x \in [0, 1]. \end{cases} \tag{4}$$

In the next exercises you will prove that s is indeed the natural cubic spline based on f. Then you will use s to approximate $f\left(-\frac{1}{2}\right)$.

(a) *Show* that s, s' and s'' are continuous on the interval $[-1, 1]$. (2 pt.)

(b) *Show* that $s''(x)$ equals zero in the end points. (1 pt.)

(c) *Show* that $s(x)$ equals $f(x)$ in the nodes. (1 pt.)

(d) *Approximate* $f\left(-\frac{1}{2}\right)$ with the use of (4). (1 pt.)

3. We derive and use Newton–Raphson's Method to solve a nonlinear problem.

(a) Given the scalar nonlinear problem:

$$\text{Find } p \in \mathbb{R} \text{ such that } f(p) = 0. \tag{5}$$

Derive Newton–Raphson's formula, given by

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \tag{6}$$

to solve the problem. Explain the method with a graph. (2pt.)

(b) Given the nonlinear problem: Find $\mathbf{p} \in \mathbb{R}^m$ such that $\mathbf{f}(\mathbf{p}) = \mathbf{0}$, where $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$. Give the Newton–Raphson's formula for this problem. (1pt.)

(c) Perform one step of the Newton–Raphson scheme applied to the following system for p_1 and p_2:

$$\begin{cases} 2p_1 - p_2 + p_1 p_2 = 0, \\ -p_1 + 2p_2 + p_2^3 = 1. \end{cases} \tag{7}$$

Use $p_1 = p_2 = 0$ as the initial estimate. (2pt.)

For the answers of this test we refer to:
https://diamhomes.ewi.tudelft.nl/~kvuik/wi3097/tentamen.html