WI4201 Written Exam
February 3rd, 2017

This examination consists of the following **5 questions**. *Read the questions* **carefully** *before answering. With each question ten credit points can be gained. This exam is an closed book exam. You are allowed a sufficiently simple calculator. You are* **not** *allowed to use any book or notes.*

Question 1 (10 pnts. - 2 pnts. per subquestion)
Answer the following questions

1. assume A to be the 2-by-2 matrix
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} .$$
Give the two Gershgorin disks that contain the eigenvalues of the matrix A;

2. give an example of a 2-by-2 non-triangular matrix A that is an M-matrix;

3. assume given an n-by-n real-valued matrix $A \in \mathbb{R}^{n \times n}$ and assume $\mathbf{u}$ to be an eigenvector of A with eigenvalue λ. Assume k to be a positive integer. Then the Krylov subspace $K^k(A, \mathbf{u})$ is a subspace in $\mathbb{R}^n$. Give the number of distinct basis requires to span this space, i.e., give the dimension of this subspace;

4. give a distinct advantage of the CGS method over the BiCG method;

5. show that the smallest eigenvalue of A and the largest eigenvalues of A^{-1} are equal.

Question 2 (10 pnts. - 2 pnt. per subquestion)

In this assignment we consider the finite difference discretization of a diffusion equation with a spatially varying diffusion coefficient on a one-dimensional domain. More specifically, given the domain $x \in \Omega = (0, 1)$, given the source function $f(x)$ and given the strictly positive diffusion coefficient $c(x) > 0 \, \forall x$, we consider finding the unknown function $u(x)$ such that $u(x)$ is a solution of the differential equation

$$-\frac{d}{dx}\left[c(x)\frac{du}{dx}\right] = f(x) \text{ for } 0 < x < 1 \text{ with } c(x) > 0 \tag{1}$$

supplied with homogeneous Dirichlet boundary conditions on both end points, i.e., we impose

$$u(0) = 0 \text{ and } u(1) = 1\,. \tag{2}$$

As an aid in answering the questions that follow, we rewrite (1) in the following equivalent form

$$-\frac{d\,\Gamma(x)}{dx} = f(x) \text{ for } 0 < x < 1 \text{ where } \Gamma(x) = c(x)\frac{du}{dx}\,. \tag{3}$$

For the finite difference discretization we consider on Ω a uniform mesh with N elements and a meshwidth $h = 1/N$. We will denote the mesh nodes as $x_i = (i-1)h$ for $1 \leq i \leq N+1$. In this way the nodes $x_1 = 0$ and $x_{N+1} = 1$ coincide with the left and right end point of Ω, respectively. We also consider the midpoints $x_{i+1/2} = [x_i + x_{i+1}]/2$ for $1 \leq i \leq N$. The finite difference discretization can be performed in two steps.

1. In the first step the derivative $d\,\Gamma/dx$ in the node $x = x_i$ can be discretized using values of $\Gamma(x)$ in the nodes $x_{i-1/2}$ and $x_{i+1/2}$;
2. In the second step the derivative du/dx in the node $x = x_{i-1/2}$ ($x_{i+1/2}$) can be discretized using values of $u(x)$ in the nodes x_{i-1} and x_i (x_i and x_{i+1}).

Let $A^h \mathbf{u}^h = \mathbf{f}^h$ denote the resulting linear system. Answer the following questions

1. give the finite difference stencil of the matrix A^h corresponding to a grid point that is neither a boundary point nor a point connected to the left or right boundary point. The element of this stencil are functions of the meshwidth h and the function $c(x)$;
2. assume that the boundary conditions are eliminated from the linear system and give the matrix A^h for $N = 3$ (and thus $h = 1/3$);
3. use the fact that $c(x) > 0 \, \forall x$ to show that the matrix A^h is symmetric and positive definite;
4. show that the method of Jacobi converges for this linear system;
5. suppose that the discretization scheme is given to be of order p. Describe a numerical test to verify that the implementation does indeed yields a discretization error of this order;

Question 3 (10 pnts. - 2 pnt. per subquestion)
Answer the following questions

1. given the linear system $A\,\mathbf{u} = \mathbf{f}$ with an n-by-n real-valued coefficient matrix A. Assume a splitting of this coefficient matrix of the form $A = M - N$ where M is non-singular and assume that a basic iterative solution method for the linear system is derived from this splitting. Derive a recursion formula for the iterands $\mathbf{u}^k$. Derive a recursion formula for the residual vector $\mathbf{r}^k$.

2. given the linear system $A\,\mathbf{u} = \mathbf{f}$ with an n-by-n real-valued coefficient matrix A. Assume the recursion formula for the error vector $\mathbf{e}^{k+1} = \left(I - M^{-1}\,A\right)\mathbf{e}^k$ to be valid. Give a sufficient condition on the matrix n-by-n $B = I - M^{-1}\,A$ for the iterative scheme to converge.

3. give the residual equations that give the relation between the error $\mathbf{e}^k$ and the residual vector $\mathbf{r}^k$. Use this relation to derive the defect-correction scheme that use the approximation $\widehat{A}$ to A;

4. assume
$$[A] = \frac{1}{h^2}\left[-1 \quad 2 - 1\right]$$
to be the stencil of the 1D Laplacian on a uniform mesh. Give stencil for the Jacobi and weighted Jacobi iteration matrix B_{JAC};

5. assume A to be SPD and let λ_1 and λ_n denote the smallest and largest eigenvalue of A. Assume $M = \tau^{-1} I$ with τ a real-valued parameter to be the splitting correspond to the Richardson method. Derive optimal value for the parameter τ.

Question 4 (10 pnts. - 2 pnt. per subquestion)

In this exercise we consider a linear system $A\mathbf{u} = \mathbf{b}$, where matrix A is real-valued and SPD. As an iterative method we use the (Preconditioned) Conjugate Gradient method.

1. What properties should be satisfied in order that a matrix A is SPD? Give a definition of the matrix A-norm;
2. Assume that our first iterate is given by $\mathbf{u}^1 = \alpha\,\mathbf{b}$. Determine an expression for the parameter α such that the error $\mathbf{u} - \mathbf{u}^1$ has a minimal length in the 2-norm;
3. We consider two 10-by-10 diagonal matrices A and C. For the first matrix we have $a_{i,i} = i,\ i = 1\ldots 10$ and for the second $c_{i,i} = 1,\ i = 1\ldots 9$ and $c_{10,10} = 1000$. For the vector $\mathbf{b}$ we have $\mathbf{b}_i = 1,\ i = 1\ldots 10$ and we take the zero vector as starting solution. For which of both systems is CG faster to converge (motivate your answer)? What is for both systems the maximum number of iterations before a solution with a sufficient small residual norm, say $\|\mathbf{r}^k\|/\|\mathbf{b}\| \leq 10^{-15}$, is obtained?
4. We now combine the CG method with a preconditioner M. Give the three properties of M in order to have a good preconditioner;
5. Explain how a preconditioner can be combined with CG? Give three possible classes of preconditioners. Compare the properties of these preconditioners in the three classes.

Question 5 (10 pnts. - 2.5 pnt. per subquestion)

In this exercise we consider the Power method to approximate the eigenvalues of an n-by-n real-valued matrix $A \in \mathbb{R}^{n \times n}$.

1. The basic Power method is given by: $\mathbf{q}_k = A\mathbf{q}_{k-1}$. The eigenvalues are ordered such that $|\lambda_1| > |\lambda_2| \geq ... \geq |\lambda_n|$. We assume that $\mathbf{q}_0$ can be written as a linear combination of the eigenvectors, with a non-zero component in the eigenvector corresponding to λ_1. Define $\lambda^{(k)} = \frac{\mathbf{q}_k^T A \mathbf{q}_k}{\|\mathbf{q}_k\|_2^2}$ and show that

$$|\lambda_1 - \lambda^{(k)}| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \ ;$$

2. Next we consider the advanced Power method:
   $\mathbf{q}_0 \in \mathbb{R}^n$ is given
   for $k = 1, 2, ...$
   $\quad \mathbf{z}_k = A\mathbf{q}_{k-1}$
   $\quad \mathbf{q}_k = \mathbf{z}_k / \|\mathbf{z}_k\|_2$
   $\quad \lambda^{(k)} = \mathbf{q}_{k-1}^T \mathbf{z}_k$
   endfor
   Show that if $\mathbf{q}_k$ is close to the eigenvector corresponding to λ_1 then $\lambda^{(k)}$ is a good approximation of λ_1;

3. Note that from part (1) it follows that the Power method is a linearly converging method. Give a good stopping criterion for the Power method;

4. Given an n-by-n real-valued matrix $A \in \mathbb{R}^{n \times n}$, where

$$\lambda_1 = 1000 \ , \quad \lambda_{n-1} = 1.1 \ \text{ and } \ \lambda_n = 1 \ .$$

   Give a fast converging variant of the Power Method to approximate λ_n.

$$\text{Total credit} = \frac{\text{credits obtained}}{5}$$