DELFT UNIVERSITY OF TECHNOLOGY
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TEST SCIENTIFIC COMPUTING (wi4201)
Friday February 1 2019, 13:30-16:30

1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) $A \in \mathbb{R}^{n\times n}$, and $A = A^T \Rightarrow \|A\| = \max\limits_{1\le i\le n} \lambda_i$ where λ_i is an eigenvalue of A. (2 pt.)

   (b) $A \in \mathbb{R}^{n\times n}$ is SPD $\Rightarrow a_{ii} > 0$. (2 pt.)

   (c) $A \in \mathbb{R}^{n\times n}$ is SPD, and $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A. Take $\mathbf{r} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$. $\Rightarrow$ the dimension of the Krylov subspace $K^{10}(A,\mathbf{r})$ is equal to 3.

   (d) $\mathbf{u} \in \mathbb{R}^n \Rightarrow \|\mathbf{u}\|_2 \le \sqrt{n}\|\mathbf{u}\|_\infty$. (2 pt.)

   (e) $A \in \mathbb{R}^{n\times n}$, $\Rightarrow \|A\|_\infty = \|A\|_1$ (2 pt.)

2. We consider the equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = f \text{ for } (x,y) \in \Omega = [0,1]\times[0,1].$$

   and Dirichlet boundary conditions on the boundary $u(x,y) = 0$, for $(x,y) \in \delta\Omega$. For the discretization we use m intervals in both coordinate directions.

   (a) Give a discretization of the equation such that the truncation error is $O(h^2)$ (+ proof). (2 pt.)

   (b) Give the stencil of this discretization in an interior node and in the lower left corner. Note that the boundary conditions are eliminated. (2 pt.)

   (c) We use lexicographic ordering of the unknowns. Give the structure of the matrix A and its bandwidth. (2 pt.)

   (d) Show that the matrix A is SPD. (2 pt.)

   (e) Which Krylov subspace method do you recommend. Include a motivation for your answer. (2 pt.)

3. (a) Given the linear system $A\mathbf{u} = \mathbf{f}$ with an n-by-n real-valued coefficient matrix A. Assume a splitting of this coefficient matrix of the form $A = M - N$ where M is non-singular and assume that a basic iterative solution method for the linear system is derived from this splitting. Derive a recursion formula for the iterates $\mathbf{u}^k$. Derive a recursion formula for the residual vector $\mathbf{r}^k$.

(2 pt.)

(b) Give the relation between the error $\mathbf{e}^k$ and the residual vector $\mathbf{r}^k$. Use this relation to derive the defect-correction scheme that use the approximation $\widehat{A}$ to A;

(2 pt.)

(c) Assume

$$[A] = \frac{1}{h^2}\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$$

to be the stencil of the 1D Laplacian on a uniform mesh. Give the stencil for the Richardson and Gauss-seidel iteration matrix B_{RIC} and B_{GS};

(2 pt.)

(d) Assume A to be SPD and let λ_1 and λ_n denote the smallest and largest eigenvalue of A. Assume $M = \tau^{-1}I$ with τ a real-valued parameter to be the splitting correspond to the Richardson method. Derive optimal value for the parameter τ. (2 pt.)

(e) Assume A to be SPD. Suppose that $\lambda_1 = 1$ and $\lambda_n = 10$. Take $\tau = \frac{1}{20}$. How many iterations k of the Richardson method are needed such that $\frac{\|\mathbf{r}_k\|_2}{\|\mathbf{r}_0\|_2} \leq 10^{-4}$? (2 pt.)

4. In this exercise we have to solve a linear system $A\mathbf{u} = \mathbf{f}$, where A is an $n \times n$ SPD matrix.

(a) Take $\mathbf{u}_1 = \alpha\mathbf{f}$. Derive an expression for α such that $\|\mathbf{f} - A\mathbf{u}_1\|_A$ is minimal. (2 pt.)

(b) Give the optimisality property of the CG method. Motivate why there is a $k \leq n$ such that $\mathbf{u}_k = \mathbf{u}$ (without rounding errors). (2 pt.)

(c) We consider two different matrices. The extreme eigenvalues of A_1 are given by $\lambda_1 = 1$ and $\lambda_n = 10$. The extreme eigenvalues of A_2 are given by $\lambda_1 = 0.1$ and $\lambda_n = 20$. For which matrix can you expect the CG method to converge faster (+ motivation)? (2 pt.)

(d) Sketch the superlinear convergence of the CG method and give a heuristic explanation of superlinear convergence. (2 pt.)

(e) For the preconditioned CG method a preconditioner matrix M is needed. Give three properties that M should satisfy. (2 pt.)

5. (a) Give the outline of the LU-decomposition method (without pivotting) to solve $A\mathbf{u} = \mathbf{f}$, where $A \in \mathbb{R}^{n \times n}$ is a non-singular matrix. Give the amount of flops for a full matrix A. (2 pt.)

   (b) Show that the inverse of the Gauss transformation $M_k = I - \alpha^{(k)} \mathbf{e}_k^T$ is the rank-one modification $M_k^{-1} = I + \alpha^{(k)} \mathbf{e}_k^T$. The k-th Gauss-vector $\alpha^{(k)} \in \mathbb{R}^n$ is defined as

$$\alpha^{(k)} = (\underbrace{0, \ldots, 0}_{k}, \underbrace{\mathbf{b}_k / a_{k,k}^{(k-1)}}_{n-k})^T . \tag{1}$$

(2 pt.)

   (c) Given the linear system $A\mathbf{u} = \mathbf{f}$ with $A \in \mathbb{R}^{n \times n}$ and the perturbed system $A(\mathbf{u} + \Delta\mathbf{u}) = \mathbf{f} + \Delta\mathbf{f}$. Derive an upperbound for $\frac{\|\Delta \mathbf{u}\|}{\|\mathbf{u}\|}$ where $\|.\|$ is an arbitrary vector norm, which has the multiplicative property. (2 pt.)

   (d) Suppose we have a penta-diagonal matrix $A \in \mathbb{R}^{n \times n}$. For a given m, where $1 < m < n$, we know that the elements $a(i-m,i)$, $a(i-1,i)$, $a(i,i)$, $a(i,i+1)$, and $a(i,i+m)$, are nonzero. Give the non-zero pattern of the L and U matrix after the LU-decomposition without pivotting. (2 pt.)

   (e) Suppose that the matrix A is SPD. Give the definition of the Cholesky decomposition. What are the advantages of the Cholesky decomposition if compared with the LU-decomposition? (2 pt.)