DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

The final grade of the test: (total number of points)/5

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TEST SCIENTIFIC COMPUTING (wi4201 / wi4201COSSE)
Wednesday January 19 2022, 13:30-16:30

1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) $A \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. $\kappa_2(A) = 1$, where κ_2 is the 2-norm condition number. (2 pt.)

   (b) the amount of work per iteration of the Bi-CGSTAB method remains constant as a function of the number of iterations; (2 pt.)

   (c) Given matrix $A \in \mathbb{R}^{n\times n}$. $P \in \mathbb{R}^{n\times n}$ is a permutation matrix. The spectra of A and P^TAP are the same. (2 pt.)

   (d) $A \in \mathbb{R}^{n\times n}$ is such that $a_{i,i} = 4, a_{i,i-1} = -1, a_{i-1,i} = -1$. All other components of A are equal to zero. $\|A\|_2 = 8$. (2 pt.)

   (e) $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ is invertible. (2 pt.)

2. Given the domain $\Omega = (0,1)\times(0,1)$ with boundary $\Gamma = \partial\Omega$, we consider the following problem:
$$-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} + cu(x,y) = f(x,y)$$
where $c > 0$, supplied with Dirichlet boundary conditions
$$u(x,y) = 1 \text{ on } \Gamma$$
For the discretization we use an equidistant grid, step size h, and lexicographic ordering of the unkowns. Please answer the following questions:

   (a) Check that the function given by:
$$u^{[k\ell]}(x,y) = \sin(k\pi x)\sin(\ell\pi y) \text{ for } k,\ell \in \mathbb{N}, k \neq 0 \text{ and } \ell \neq 0$$
   is an eigenfunction of operator $-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} + cu(x,y)$, and give an expression for the eigenvalue.
   (2.5 pt.)

(b) For an internal grid point the stencil is given by:

$$\frac{1}{h^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4+ch^2 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Show that the numerical method has a local truncation error of $O(h^2)$. (2.5 pt.)

(c) Give the stencil and right-hand side for the grid point located at $(x, y) = (1 - h, 1 - h)$, with elimination of the boundary conditions. (2.5 pt.)

(d) Show that the corresponding coefficient matrix A is positive definite. (2.5 pt.)

3. (a) Given the linear system $A\mathbf{u} = \mathbf{f}$ with $A \in \mathbb{R}^{n\times n}$. Consider a splitting of the form $A = M - N$ where M is non-singular. Derive a recursion formula for the BIM (Basic Iterative Method) iterates $\mathbf{u}^k$. Derive a recursion formula for the residual vector $\mathbf{r}^k$. (2 pt.)

(b) Give the iteration matrix B for this BIM and give a sufficient condition such that the BIM converges. (2 pt.)

(c) Suppose A is a non-singular lower triangular matrix. Show that the Gauss Seidel method converges for such a matrix. (2 pt.)

(d) Suppose A is a non-singular lower triangular matrix. Show that the Jacobi method converges for such a matrix. (2 pt.)

(e) Give three different stopping criteria and specify the good and bad properties of these stopping criteria. (2 pt.)

4. Consider the linear system $A\mathbf{u} = \mathbf{f}$, where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix.

(a) If A is SPD show that $<\mathbf{y}, \mathbf{z}>_A = \mathbf{y}^T A\mathbf{z}$ is an inner product.
(Hint: an inner product has the following properties: $<\mathbf{u}, \mathbf{v}> = <\mathbf{v}, \mathbf{u}>$, $<c\,\mathbf{u}, \mathbf{v}> = c <\mathbf{u}, \mathbf{v}>$, $<\mathbf{u}+\mathbf{v}, \mathbf{w}> = <\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}>$ and $<\mathbf{u}, \mathbf{u}> \geq 0$ with equality only for $\mathbf{u} = 0$) (2 pt.)

(b) We assume that $\mathbf{u}^1 = \alpha_0 \mathbf{r}^0$. Determine α_0 such that $\|\mathbf{u} - \mathbf{u}^1\|_A$ is minimal. (2 pt.)

(c) The matrix A corresponds to a shifted discretized Poisson operator. The eigenvalues are given by

$$\lambda_{k,\ell} = 6 - 2\cos\frac{\pi k}{61} - 2\cos\frac{\pi \ell}{61} \quad , \quad 1 \leq k, \ell \leq 60.$$

Determine the linear rate of convergence for the Conjugate Gradient method. (2 pt.)

(d) If the convergence of the Conjugate Gradient method is too slow a preconditioner M could be used. Give three properties for matrix M in orde to be a suitable preconditioner. How can such a preconditioner be combined with Conjugate Gradient in order to obtain the Preconditioned Conjugate Gradient method? (2 pt.)

(e) Consider matrix A given by: $A = \begin{pmatrix} 100 & -1 & 0 \\ -1 & 100 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Give an estimate of the convergence of CG. Take for the preconditioner the best diagonal matrix and give an estimate of the convergence of PCG. (2 pt.)

5. Consider the Power method to approximate the eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$. The eigenvalues are ordered such that $|\lambda_1| > |\lambda_2| \geq ... \geq |\lambda_n|$ and $\lambda_1 \in \mathbb{R}$.

(a) The basic Power method is given by: $\mathbf{q}_k = A\mathbf{q}_{k-1}$. We assume that $\mathbf{q}_0$ can be written as a linear combination of the eigenvectors, with a non-zero component of the eigenvector corresponding to λ_1. Define $\lambda^{(k)} = \frac{\mathbf{q}_k^T A \mathbf{q}_k}{\|\mathbf{q}_k\|_2^2}$ and show that

$$|\lambda_1 - \lambda^{(k)}| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) .$$

(2.5 pt.)

(b) Now we consider the advanced Power method:
$\mathbf{q}_0 \in \mathbb{R}^n$ is given
for $k = 1, 2, ...$
$\quad \mathbf{z}_k = A\mathbf{q}_{k-1}$
$\quad \mathbf{q}_k = \mathbf{z}_k / \|\mathbf{z}_k\|_2$
$\quad \lambda^{(k)} = \bar{\mathbf{q}}_{k-1}^T \mathbf{z}_k$
endfor
Show that if $\mathbf{q}_{k-1}$ is close to the eigenvector corresponding to λ_1 then $\lambda^{(k)}$ is a good approximation of λ_1. (2.5 pt.)

(c) Note that from part (a) it follows that the Power method is a linearly converging method. Give a stopping criterion for the Power method. (2.5 pt.)

(d) Given a matrix $A \in \mathbb{R}^{n \times n}$, where

$$\lambda_1 = 2000 , \quad \lambda_{n-1} = 2.1 \text{ and } \lambda_n = 2 .$$

Let σ be a shift value larger than λ_n. Give the shift and invert Power method to approximate λ_n. Give the rate of convergence for this method. (2.5 pt.)