Extending the Method of Fundamental Solutions to Non Homogeneous Elastic Wave Propagation Problems

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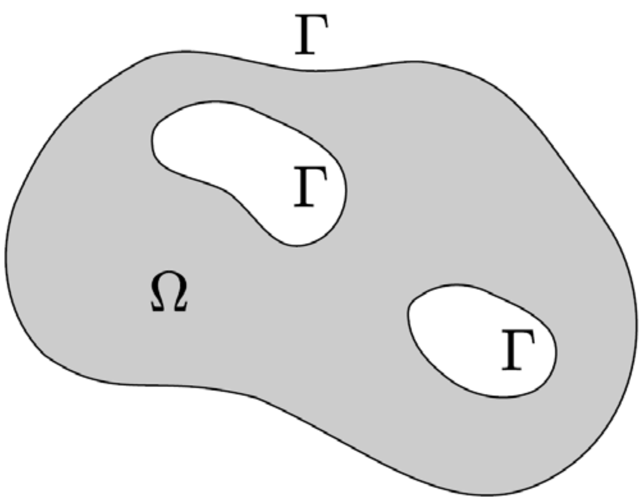
Outline:

- Elastic Wave Propagation Problems
- The Classical Method of Fundamental Solutions (MFS)
  - Motivation and numerical formulation.
  - Theoretical and numerical results
- Extending the MFS to Non Homogeneous BVPs
- Numerical Examples
  - PDE with constant frequency
  - Interior wave scattering problem
  - A more general PDE (variable coefficients)

• Elastic Wave Propagation Problems



$$\begin{cases} \mathcal{E}\boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma \end{cases}$$

- Continuous, isotropic elastic medium $\Omega \subset \mathbb{R}^d$
- Cauchy-Navier equations of elastodynamics:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} = \rho \ddot{u}_i, \quad i = 1, \dots, d.$$

- Search for a time-harmonic solution:

$$\boldsymbol{u}(x,t) = \boldsymbol{u}(x) e^{-i\omega t}$$

$$\mathcal{E}\boldsymbol{u} := \mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{u}) + \rho \omega^2 \boldsymbol{u} = 0$$

ρ - density, λ, μ - Lamé constants, ω - frequency

Compressional wave number $k_p = \omega \sqrt{\frac{\rho}{\lambda + 2\mu}}$ and shear wave number $k_s = \omega \sqrt{\frac{\rho}{\mu}}$

Kupradze tensor (FS)

$$\mathbb{G}_\omega(x) = \frac{1}{\rho \omega^2} \left[k_s^2 \Phi_{k_s}(x) \mathbb{I} + \mathbb{D}(\Phi_{k_s} - \Phi_{k_p})(x) \right]$$

$\mathbb{I} = \delta_{ij}$

$\mathbb{D} = \partial_{ij}$

where Φ_{k_p} and Φ_{k_s} are FS for the Helmholtz operators with frequencies k_p and k_s

• Motivation for the Method of Fundamental Solutions (MFS)

- Consider the single layer potential (s.l.p.) for the solution of the Dirichlet BVP

$$\boldsymbol{u}(x) = (\boldsymbol{\mathcal{L}}\boldsymbol{\varphi})(x) = \int_\Gamma \mathbb{G}_\omega(x-y)\boldsymbol{\varphi}(y)\,\mathrm{d}s_y, \quad x \in \Omega$$

(continuous across Γ)

- Define a Fredholm BIE of the first kind for the density $\boldsymbol{\varphi}$

$$\int_\Gamma \mathbb{G}_\omega(x-y)\boldsymbol{\varphi}(y)\,\mathrm{d}s_y = \boldsymbol{g}(x), \quad x \in \Gamma$$

(an ill-posed problem)
(the kernel is singular)

- Consider the s.l.p of $\boldsymbol{u}$ on an auxiliary boundary $\hat{\Gamma} = \partial\hat{\Omega}$ with $\hat{\Omega} \supset \bar{\Omega}$

$$\boldsymbol{u}(x) = (\hat{\boldsymbol{\mathcal{L}}}\boldsymbol{\varphi})(x) = \int_{\hat{\Gamma}} \mathbb{G}_\omega(x-y)\boldsymbol{\varphi}(y)\,\mathrm{d}s_y, \quad x \in \bar{\Omega}$$

(regular integral)

- Approximate the integral by a quadrature rule with weights γ_j and knots $y_j \in \hat{\Gamma}$

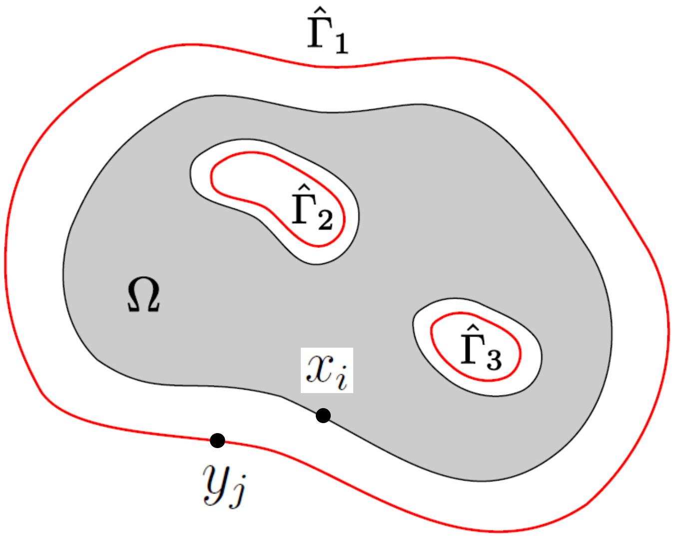
$$\boldsymbol{u}(x) \approx \tilde{\boldsymbol{u}}(x) = \sum_j \gamma_j \mathbb{G}_\omega(x-y_j)\boldsymbol{\varphi}(y_j)$$

($\tilde{u}$ satisfies the PDE)

- The approximate solution of the BIE $\hat{\boldsymbol{\mathcal{L}}}\boldsymbol{\varphi} = \boldsymbol{g}$ is reduced to the problem:

Find coefficients $\boldsymbol{\varphi}(y_j) \in \mathbb{C}^d$, such that $\tilde{\boldsymbol{u}}$ satisfies (approximately) the BC

• The Classical Method of Fundamental Solutions



$\mathcal{Y} = \{y_j \in \hat{\Gamma} : \; j = 1, \ldots, n\}$ source points

$$\boxed{\tilde{\boldsymbol{u}}(x) = \sum_{j=1}^{n} \mathbb{G}_\omega(x - y_j) \cdot \mathbf{a}_j} \quad \mathbf{a}_j \in \mathbb{C}^d, \;\; x \in \bar{\Omega}$$

$\mathcal{X} = \{x_i \in \Gamma : \; i = 1, \ldots, m\}$ collocation points

$$\boxed{\tilde{\boldsymbol{u}}(x_i) = \boldsymbol{g}(x_i), \;\; x_i \in \mathcal{X}}$$

$$\overbrace{\begin{bmatrix} \mathbb{G}_\omega(x_1 - y_1) & \cdots & \mathbb{G}_\omega(x_1 - y_n) \\ \vdots & \ddots & \vdots \\ \mathbb{G}_\omega(x_m - y_1) & \cdots & \mathbb{G}_\omega(x_m - y_n) \end{bmatrix}}^{\boldsymbol{B}(\omega, \mathcal{X}, \mathcal{Y})} \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}(x_1) \\ \vdots \\ \boldsymbol{g}(x_m) \end{bmatrix}$$

$(md) \times (nd)$

Linear System

- Solving the Linear System: **collocation** (if $n = m$) or **least squares** (if $n > m$)
- Regularization is required: Truncates Singular Value Decomposition (**TSVD**)

• Theoretical Results (homogeneous BVP)

$$\boldsymbol{\mathcal{S}}_D(\Gamma,\hat{\Gamma}) = \mathrm{span}\{\mathbb{G}_\omega(x-y)|_{x\in\Gamma} : \ y \in \hat{\Gamma}\}$$

MFS approximation space

- Assume that $\omega > 0$ is not an eigenfrequency for the Dirichlet BVP in Ω

Lemma: The restrictions to Γ of $\mathbb{G}_\omega(\cdot - y_1), \ldots, \mathbb{G}_\omega(\cdot - y_n)$ are linearly independent.

Theorem [Density result]: For $r \geq 1/2$ the space $\boldsymbol{\mathcal{S}}_D(\Gamma,\hat{\Gamma})$ is dense in $[H^r(\Gamma)]^d$.

Proof:

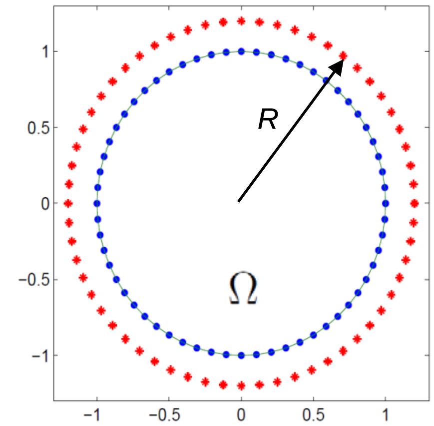
$$\hat{\boldsymbol{\mathcal{L}}} : [H^{r-1}(\hat{\Gamma})]^d \to [H^r(\Gamma)]^d \qquad \xrightarrow{\langle \hat{\mathcal{L}}\varphi, \phi\rangle = \langle \varphi, \hat{\mathcal{L}}^*\phi\rangle} \qquad \hat{\boldsymbol{\mathcal{L}}}^* : [H^{-r}(\Gamma)]^d \to [H^{-r+1}(\hat{\Gamma})]^d$$

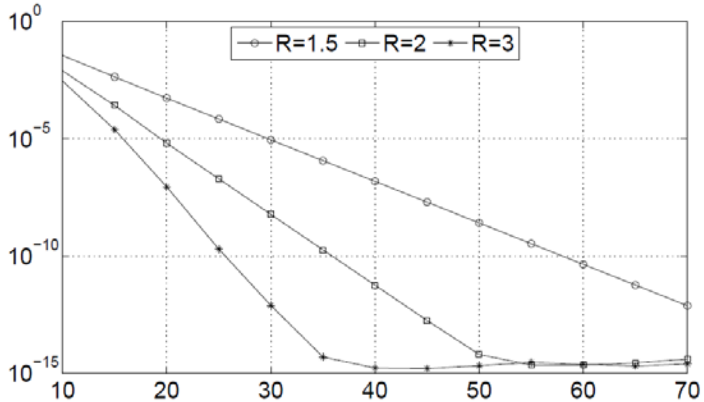
$$(\hat{\boldsymbol{\mathcal{L}}}\boldsymbol{\varphi})(x) = \int_{\hat{\Gamma}} \mathbb{G}_\omega(x-y)\boldsymbol{\varphi}(y)\,\mathrm{d}s_y \qquad (\hat{\boldsymbol{\mathcal{L}}}^*\boldsymbol{\phi})(y) = \int_{\Gamma} \overline{\mathbb{G}}_\omega(x-y)\boldsymbol{\phi}(x)\,\mathrm{d}s_x$$

- $\boldsymbol{\mathcal{S}}_D(\Gamma,\hat{\Gamma})$ is dense in $\mathcal{R}(\hat{\boldsymbol{\mathcal{L}}})$ (discretization argument);
- $[\mathcal{R}(\hat{\boldsymbol{\mathcal{L}}})]^\perp = ker(\hat{\boldsymbol{\mathcal{L}}}^*)$ (bounded linear operators acting between Banach spaces);
- it is sufficient to show that $ker(\hat{\boldsymbol{\mathcal{L}}}^*) = \{0\}$.

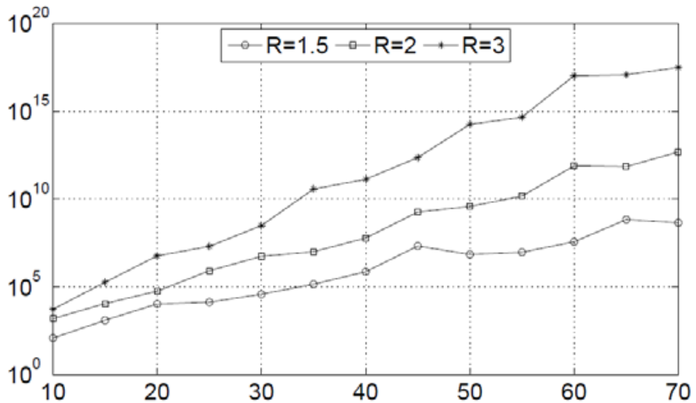
• Typical Numerical Behavior of the MFS (*circular domain*)

- $\Omega = B(0,1) \subset \mathbb{R}^2,\ \hat{\Gamma} = S^1_R\ (R > 1),\ \lambda = 1,\ \mu = 2,\ \rho = 1,\ \omega = 1$
- $\boldsymbol{g}(x) = \boldsymbol{d}e^{ik_p x\cdot \boldsymbol{d}} + \boldsymbol{d}^{\perp}e^{ik_s x\cdot \boldsymbol{d}}$ with $\boldsymbol{d} = (1,1)/\sqrt{2}$ (P-wave & S-wave)
- Remark: choose $n = m$ for higher accuracy (smooth settings)

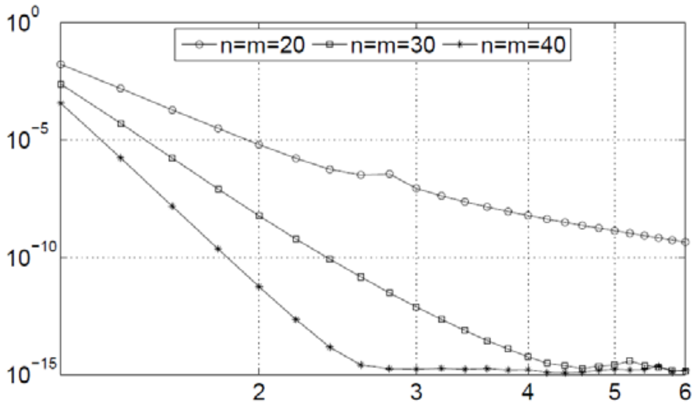




ε_∞ as a function of $n = m$ (semi-log)



$Cond_2(\boldsymbol{B})$ as a function of $n = m$ (semi-log)



ε_∞ as a function of R (log-log)

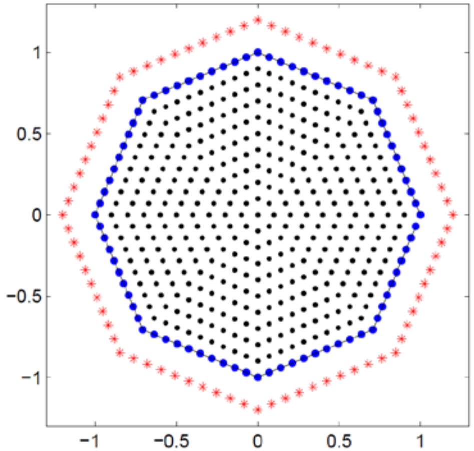
- Exponential convergence with $n = m$
- Algebraic convergence with R
- Trade-off between accuracy and conditioning

Non Homogeneous PDE

• The MFS for nonhomogeneous PDEs

$$\begin{cases} \Delta^\star \boldsymbol{u} + \rho\omega^2 \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma \end{cases}$$

$$\tilde{\boldsymbol{u}}(x) = \sum_{r=1}^{p} \sum_{j=1}^{n} \mathbb{G}_{\omega_r}(x - y_j) \cdot \mathbf{a}_{r,j} \qquad \mathbf{a}_{r,j} \in \mathbb{C}^d, \; x \in \bar{\Omega}$$



$\mathcal{Y} = \{y_j \in \hat{\Gamma} : j = 1, \ldots, n\}$

$\mathcal{W} = \{\omega_r > 0 : r = 1, \ldots, p\}$

DOF

$\mathcal{X}_0 = \{x_i \in \Sigma : i = 1, \ldots, m_0, \; \bar{\Omega} \subseteq \Sigma \subset \hat{\Omega}\}$

$\mathcal{X}_1 = \{x_i \in \Gamma : i = 1, \ldots, m_1\}$

collocation points

$$\begin{bmatrix} \rho(\omega^2 - \omega_1^2)\boldsymbol{B}(\omega_1, \mathcal{X}_0, \mathcal{Y}) & \ldots & \rho(\omega^2 - \omega_p^2)\boldsymbol{B}(\omega_p, \mathcal{X}_0, \mathcal{Y}) \\ \boldsymbol{B}(\omega_1, \mathcal{X}_1, \mathcal{Y}) & \ldots & \boldsymbol{B}(\omega_p, \mathcal{X}_1, \mathcal{Y}) \end{bmatrix} \begin{bmatrix} \mathbf{a}_{1,1} \\ \vdots \\ \mathbf{a}_{p,n} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathcal{X}_0) \\ \mathbf{g}(\mathcal{X}_1) \end{bmatrix}$$

Linear system $[d(m_0 + m_1)] \times [dnp]$

† Note that $(\Delta^\star + \rho\omega^2)\mathbb{G}_{\omega_r} = \rho(\omega^2 - \omega_r^2)\mathbb{G}_{\omega_r}$ since $\Delta^\star \mathbb{G}_{\omega_r} = -\omega_r^2 \mathbb{G}_{\omega_r}$ (no differentiation)

- Solving the Linear System: **collocation** (if $n = m$) or **least squares** (if $n > m$)
- Regularization is required: Truncates Singular Value Decomposition (**TSVD**)

• Theoretical Results (density result and error bound)

Theorem [Density result]: $\boldsymbol{\mathcal{S}} = \text{span}\{\mathbb{G}_\omega(x-y)|_{x\in\Omega} \,:\, y \in \hat{\Gamma},\ \omega \in \mathbb{R}^+\}$ is dense in $[L^2(\Omega)]^d$.

$$\tilde{\boldsymbol{f}}(x) = \sum_{r=1}^{p}\sum_{j=1}^{n} \mathbb{G}_{\omega_r}(x-y_j)\cdot \mathbf{a}_{r,j}, \quad x \in \bar{\Omega}$$ such that $$||\boldsymbol{f} - \tilde{\boldsymbol{f}}||_{[L^2(\Omega)]^d} \le \varepsilon_1$$

$$\tilde{\boldsymbol{u}}_P(x) = \sum_{r=1}^{p}\sum_{j=1}^{n} \frac{1}{\rho(\omega^2-\omega_r^2)}\mathbb{G}_{\omega_r}(x-y_j)\cdot \mathbf{a}_{r,j}, \quad x \in \bar{\Omega}$$ satisfies $$\left(\Delta^\star + \rho\omega^2\right)\tilde{\boldsymbol{u}}_P = \tilde{\boldsymbol{f}}$$

$$\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_H \quad \approx \quad \tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_P + \tilde{\boldsymbol{u}}_H$$

$$\begin{cases} \left(\Delta^\star + \rho\omega^2\right)\boldsymbol{u}_P = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u}_P = \tilde{\boldsymbol{u}}_P & \text{on } \Gamma \end{cases}$$

$$\begin{cases} \left(\Delta^\star + \rho\omega^2\right)\boldsymbol{u}_H = 0 & \text{in } \Omega \\ \boldsymbol{u}_H = \boldsymbol{g} - \tilde{\boldsymbol{u}}_P & \text{on } \Gamma \end{cases}$$

well posedness

$$||\boldsymbol{u}_P - \tilde{\boldsymbol{u}}_P||_{[H^1(\Omega)]^d} \le C_1\varepsilon_1$$

classical MFS

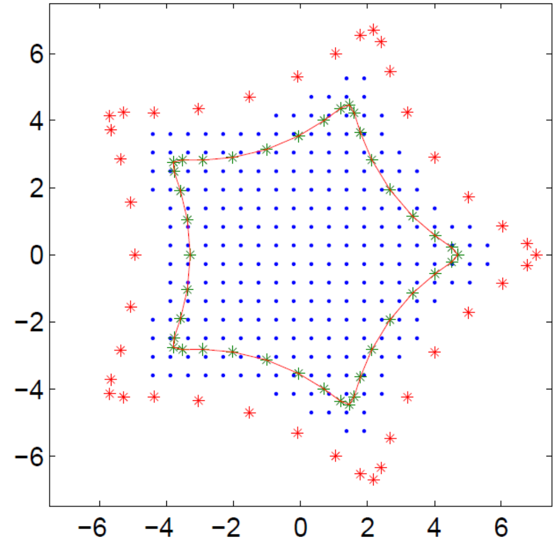
$$||\boldsymbol{u}_H - \tilde{\boldsymbol{u}}_H||_{[H^1(\Omega)]^d} \le \varepsilon_2$$

Choose ε_1 and ε_2 such that $C_1\varepsilon_1 + \varepsilon_2 \le \varepsilon$

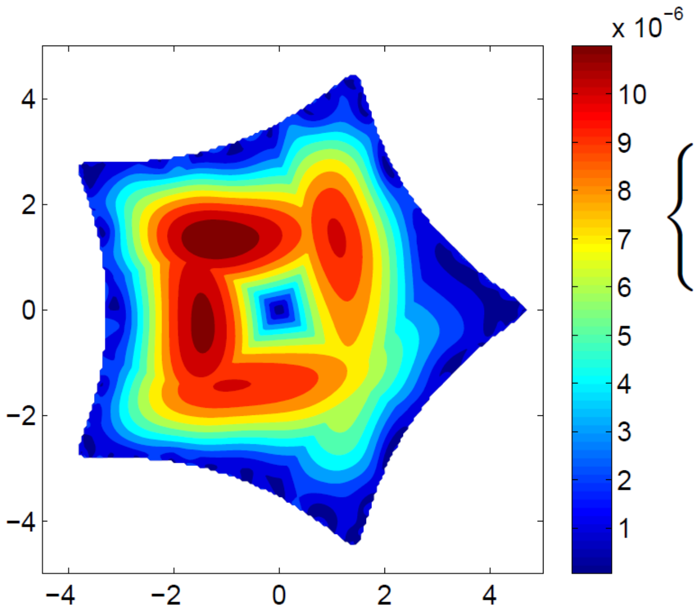
$$||\boldsymbol{u} - \tilde{\boldsymbol{u}}||_{[H^1(\Omega)]^d} \le ||\boldsymbol{u}_P - \tilde{\boldsymbol{u}}_P||_{[H^1(\Omega)]^d} + ||\boldsymbol{u}_H - \tilde{\boldsymbol{u}}_H||_{[H^1(\Omega)]^d} \le \varepsilon$$

• Numerical Example #1 (*known exact solution*)

- Parametric domain: $z(t) = 4e^{it} + 0.7e^{-4it}, \; t \in [0, 2\pi];$
- $\lambda = 2, \; \mu = 2, \; \omega = 2, \; \rho = 1;$
- $\boldsymbol{u}(x) = \left\{ \begin{array}{c} i\cos(x_1 - x_2) - \sin(x_1 - x_2) \\ i\exp(-x_1^2) + \cos(x_1 + x_2)\} \end{array} \right\};$
- $\hat{\Gamma} = 2.5 \times \Gamma, \; n = 40, \; \Sigma = 1.3 \times \bar{\Omega}, \; m_0 = 974, \; m_1 = 140.$



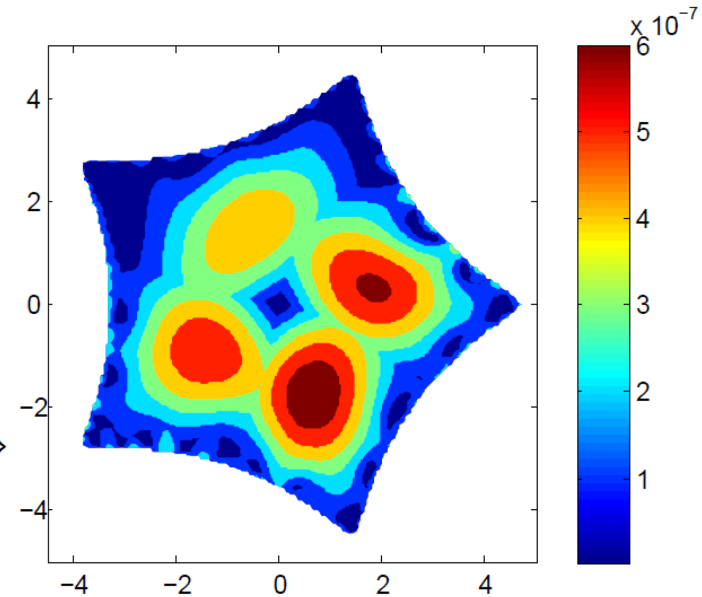
RMS error



$\mathcal{W} = \{\sqrt{i^2 + j^2} : i, j = 1, \ldots, 6\}$

$\left\{ \begin{array}{l} \Gamma : 1.61 \times 10^{-6} \\ \Omega : 3.16 \times 10^{-6} \end{array} \right.$

$\left. \begin{array}{l} \Gamma : 1.73 \times 10^{-7} \\ \Omega : 8.07 \times 10^{-7} \end{array} \right\}$



$\mathcal{W} = \{0.5 : 0.5 : 10.5\}$

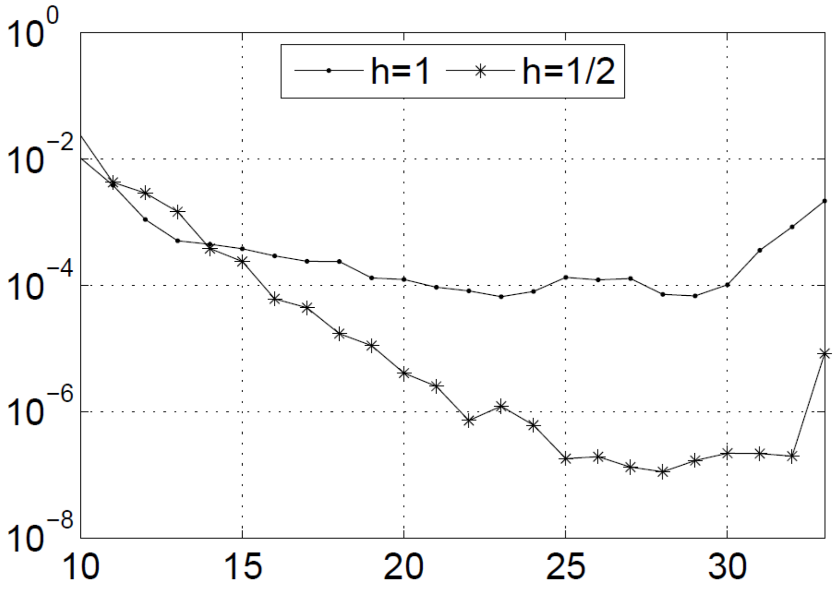
$p = 21$

• Numerical Example #1 (*cont.*) – Convergence

- Error decreases with the number of collocation points $(m_0,\ m_1)$
- Error decreases with the number of source points (n)
- Error decreases with the distance between Γ and $\hat{\Gamma}$
- Condition number of the linear system increases with $m_0,\ m_1,\ n$ and p

Like in MFS

- Error decreases with the number of test frequencies (p)



The error $\varepsilon_\infty^\Omega$ as a function of $\#\mathcal{W}$

- The choice of $\mathcal{W}$ is of the utmost importance

$$\mathcal{W}_1 = \{i\ :\ i = 1, 2, \ldots, p\}$$

$$\mathcal{W}_{1/2} = \{i/2\ :\ i = 1, 2, \ldots, p\}$$

Highly accurate numerical results may be achieved only by varying all the parameters simultaneously. Parameters $\hat{\Gamma},\ \Sigma,\ m_0,\ m_1,\ n,\ p$ are interdependent.

• Numerical Example #1 (*cont.*) – Higher Accuracy

PDE residuals $R_i^{\Omega} = f(x_i) - (\Delta^{\star} + \rho\omega^2)\tilde{u}(x_i), \quad x_i \in \mathcal{X}_0$

BC residuals $R_i^{\Gamma} = g(x_i) - \tilde{u}(x_i), \quad x_i \in \mathcal{X}_1$

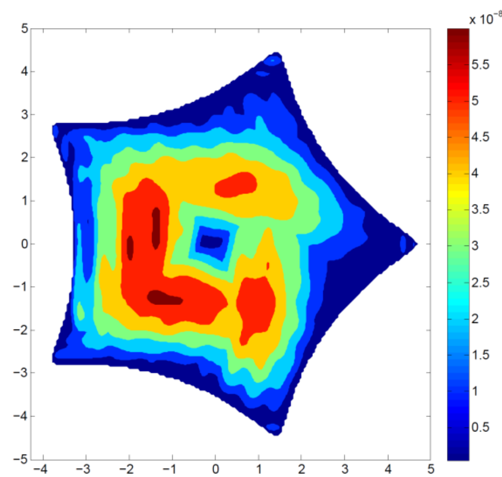
Least squares functional $J = \frac{1}{2}\left(\sum_{i=1}^{m_0}\left[R_i^{\Omega}\right]^2 + \alpha \sum_{i=1}^{m_1}\left[R_i^{\Gamma}\right]^2\right)$ α - penalty coefficient

α - relative weight of the boundary residuals with respect to the interior residuals

$$\begin{bmatrix} \rho(\omega^2-\omega_1^2)\boldsymbol{B}(\omega_1,\mathcal{X}_0,\mathcal{Y}) & \dots & \rho(\omega^2-\omega_p^2)\boldsymbol{B}(\omega_p,\mathcal{X}_0,\mathcal{Y}) \\ \alpha\ \boldsymbol{B}(\omega_1,\mathcal{X}_1,\mathcal{Y}) & \dots & \alpha\ \boldsymbol{B}(\omega_p,\mathcal{X}_1,\mathcal{Y}) \end{bmatrix} \begin{bmatrix} \mathbf{a}_{1,1} \\ \vdots \\ \mathbf{a}_{p,n} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathcal{X}_0) \\ \alpha\ \mathbf{g}(\mathcal{X}_1) \end{bmatrix}$$

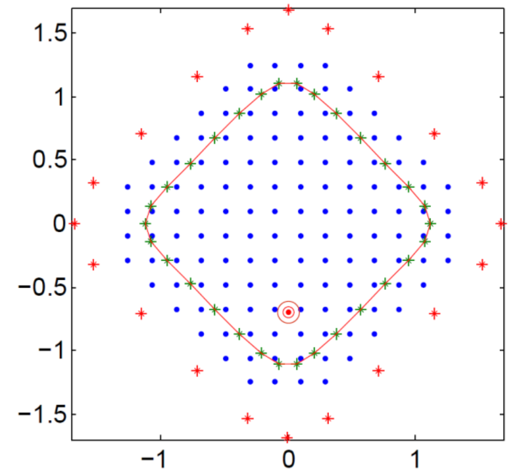
Least squares
+
Regularization

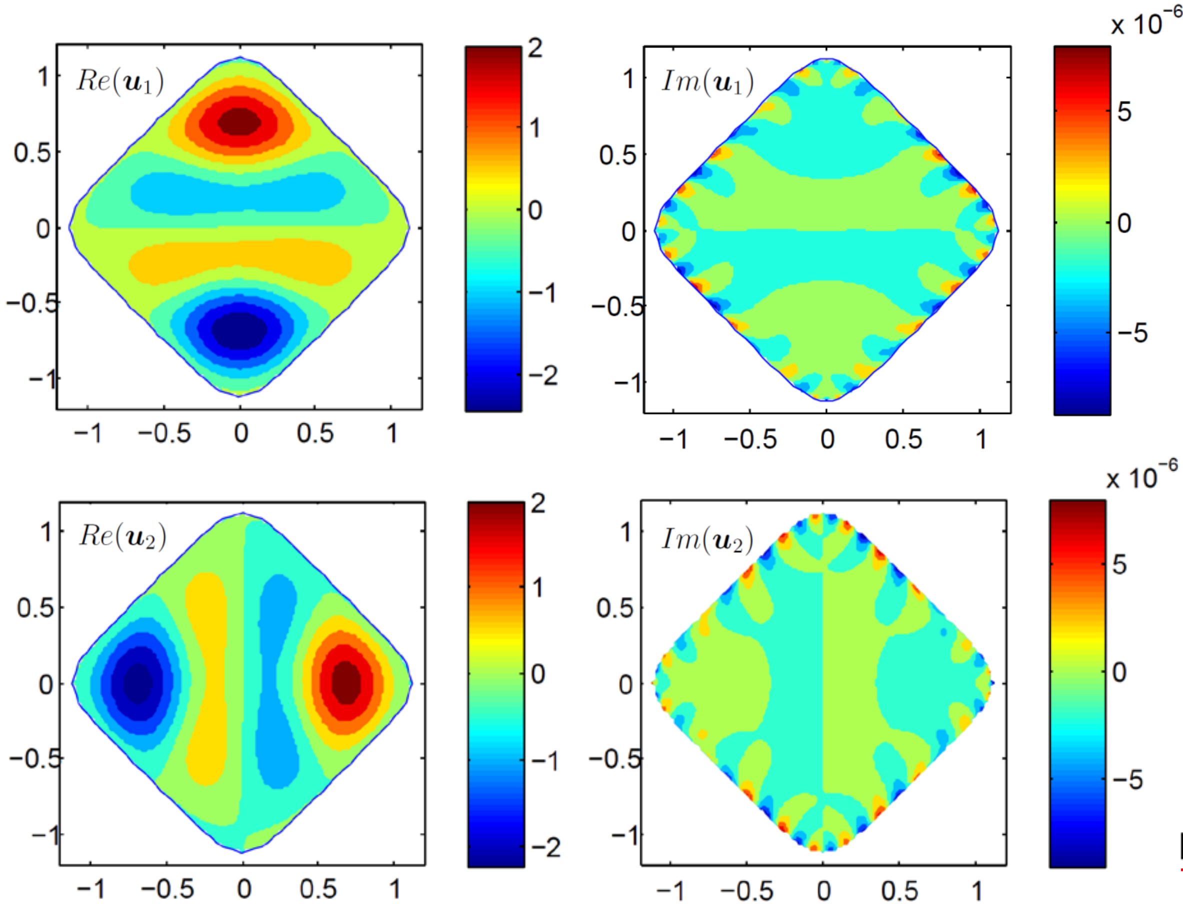
α	$\varepsilon_{\infty}^{\Omega}$	$\varepsilon_{\infty}^{\Gamma}$	$\varepsilon_{\infty}^{PDE}$
1	6.8073×10^{-7}	3.6926×10^{-7}	8.8277×10^{-6}
50	1.7285×10^{-7}	1.6129×10^{-8}	7.9281×10^{-6}
100	8.5050×10^{-8}	8.4475×10^{-9}	6.6461×10^{-6}
150	6.2990×10^{-8}	6.0600×10^{-9}	5.6295×10^{-6}
200	3.5755×10^{-7}	4.4197×10^{-9}	6.6751×10^{-6}



• Numerical Example #2 (*interior wave scattering*)

- $\boldsymbol{g}(x) = -\boldsymbol{u}^{inc}(x) = -Re[\mathbb{G}_\omega(x - S)\boldsymbol{e}_1]$ with $S = (0, -0.7)$;
- $\boldsymbol{f}(x) = \{\sin(x_1 + x_2)\,;\, \cos(x_1 + x_2)\}$;
- $\lambda = 2,\ \mu = 2,\ \rho = 1,\ \omega = 10$;
- $\hat{\Gamma} = 2.5 \times \Gamma$, $n = 50$, $\Sigma = 1.3 \times \bar{\Omega}$, $m_0 = 1228$, $m_1 = 300$, $p = 28$.





- Linear system: 3056×2800
- Condition number: $\sim 10^{30}$
- TSVD residual error: 1.75×10^{-4}

RMS error

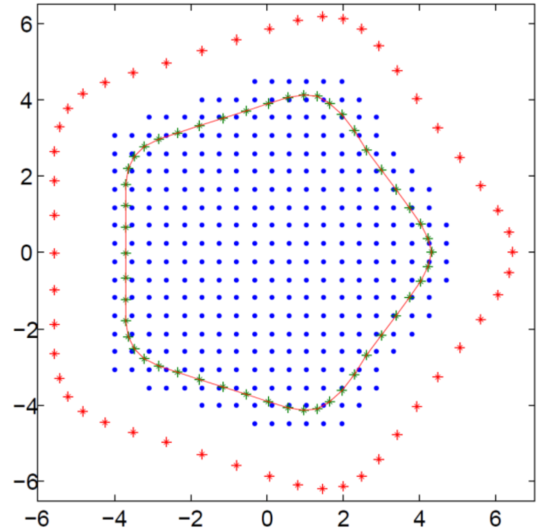
- $||\boldsymbol{g} - \tilde{\boldsymbol{u}}||_\Gamma = 7.1377 \times 10^{-6}$
- $||\boldsymbol{f} - \mathcal{E}\tilde{\boldsymbol{u}}||_\Omega = 3.9820 \times 10^{-7}$

Remark: $Im(\tilde{\boldsymbol{u}}) \sim 10^{-6}$ (neglectable)

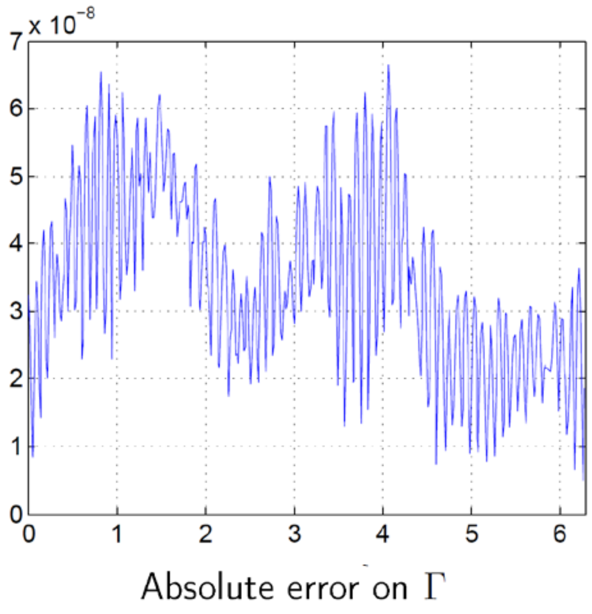
• Numerical Example #3 (*more general PDEs*)

$$\begin{cases} \Delta^\star \boldsymbol{u} + a\,\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma \end{cases}$$

$$a(x) = 2 + \sin(x_1 + x_2)$$



- $\lambda = 2, \;\mu = 2;$
- $\boldsymbol{u}(x) = \begin{Bmatrix} i\cos(x_1 - x_2) - \sin(x_1 - x_2) \\ i\exp(-x_1^2) + \cos(x_1 + x_2)\} \end{Bmatrix};$
- $\hat{\Gamma} = 1.5 \times \Gamma,\; n = 50,\; \Sigma = 1.2 \times \bar{\Omega},\; m_0 = 1910,\; m_1 = 150;$
- $\mathcal{W} = \{0.5 : 0.5 : 12\},\; p = 24.$

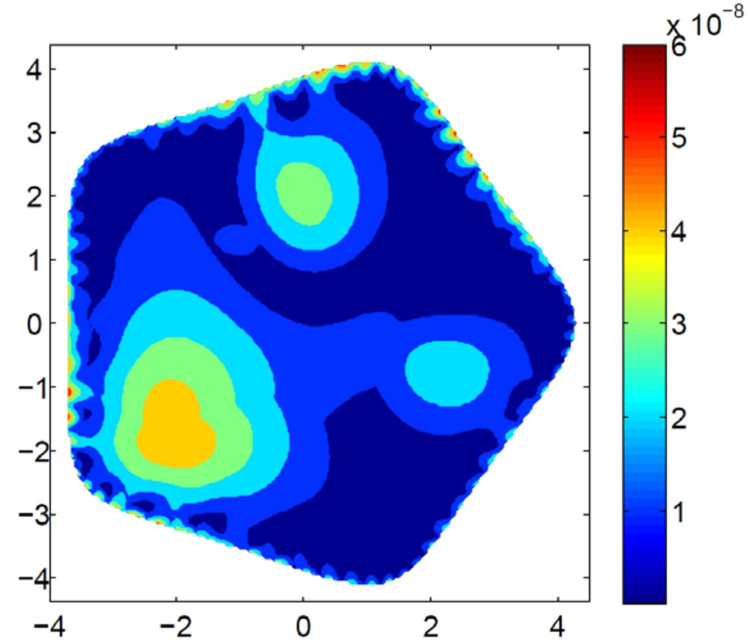


Absolute error on Γ

- Linear system: 4120×2400
- Condition number: $\sim 10^{18}$
- TSVD residual error: 1.65×10^{-6}

<u>RMS error</u>

- $||\boldsymbol{g} - \tilde{\boldsymbol{u}}||_\Gamma = 3.0631 \times 10^{-8}$
- $||\boldsymbol{f} - \mathcal{E}\tilde{\boldsymbol{u}}||_\Omega = 2.2899 \times 10^{-8}$
- $||\boldsymbol{g} - \tilde{\boldsymbol{u}}||_\Omega = 6.1813 \times 10^{-8}$



• Publications

- Carlos J. S. Alves, Nuno F. M. Martins and Svilen S. Valtchev, Extending the method of fundamental solutions to non homogeneous elastic wave problems, submitted, 2015.
- Pedro R. S. Antunes, Svilen S. Valtchev, A meshfree numerical method for acoustic wave propagation problems in planar domains with corners and cracks, J Comput. Appl. Math, 234, pp. 2646-2662, 2010.
- Svilen S. Valtchev, Asymptotic analysis of the method of fundamental solutions for acoustic wave propagation, Numerical Analysis and Applied Mathematics, AIP Conference Proceedings, vol. 1281, pp. 1179-1182, 2010.
- Svilen S. Valtchev, Nilson C. Roberty, A time-marching MFS scheme for heat conduction problems, Eng. Analysis Bound. Elements, 32, pp. 480-493, 2008.
- Carlos J. S. Alves, Svilen S. Valtchev, A Kansa Type Method Using Fundamental Solutions Applied to Elliptic PDEs, Advances in Meshfree Techniques, Computational Methods in Applied Sciences, vol. 5, Springer, 2006.
- Carlos J. S. Alves, Svilen S. Valtchev, Numerical comparison of two meshfree methods for acoustic wave scattering, Eng. Analysis Bound. Elements, 29, pp. 371-382, 2005.