A monotonicity property of the power function of multivariate tests

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Abstract

Let $S = \sum_{k=1}^n X_k X_k'$, where the X_k are independent observations from a 2-dimensional normal $N(\mu_k, \Sigma)$ distribution, and let $\Lambda = \sum_{k=1}^n \mu_k \mu_k' \Sigma^{-1}$ be a diagonal matrix of the form λI, where $\lambda \geq 0$ and I is the identity matrix. It is shown that the density ϕ of the vector $\tilde{\ell} = (\ell_1, \ell_2)$ of characteristic roots of S can be written as $G(\lambda, \ell_1, \ell_2)\phi_0(\tilde{\ell})$, where G satisfies the FKG condition on $\mathbb{R}_+^3$. This implies that the power function of tests with monotone acceptance region in ℓ_1 and ℓ_2, i.e. a region of the form $\{g(\ell_1, \ell_2) \leq c\}$, where g is nondecreasing in each argument, is nondecreasing in λ. It is also shown that the density ϕ of (ℓ_1, ℓ_2) does not allow a decomposition $\phi(\ell_1, \ell_2) = G(\lambda, \ell_1, \ell_2)\phi_0(\tilde{\ell})$, with G satisfying the FKG condition, if $\Lambda = \text{diag}(\lambda, 0)$ and $\lambda > 0$, implying that this approach to proving monotonicity of the power function fails in general.

Key words and phrases: monotonicity of power functions, noncentral Wishart matrix, characteristic roots, orthogonal groups, Euler angles, correlation inequalities, hypergeometric functions of matrix arguments, FKG inequality, pairwise total positive of order two.

1 Introduction

Let X be a normally distributed random $p \times n$ matrix with expectation $EX = \mu$ and independent columns with common covariance matrix Σ. Here and in the sequel we assume $n \geq p$. Let $\tilde{\ell}$ denote the vector of characteristic roots of XX' and let $\tilde{\lambda}$ denote the vector of characteristic roots of the noncentrality matrix $\mu\mu'\Sigma^{-1}$. It is shown in Perlman and Olkin (1980) that any test of the hypothesis $\mu = 0$ versus $\mu \neq 0$ with acceptance region $\{g(\tilde{\ell}) \leq c\}$, where g is nondecreasing in each argument, is unbiased. Furthermore they make the conjecture that the power function of such a test is nondecreasing in each component λ_i of the vector of noncentrality parameters $\tilde{\lambda}$ and suggest that this result could be proved by showing that the density of ϕ of $\tilde{\ell}$ can be written $\phi(\tilde{\ell}) = G(\tilde{\lambda}\tilde{\ell})\phi_0(\tilde{\ell})$, where G is pairwise TP_2 (totally positive of order 2) in the pairs $(\ell_i, \ell_j), i \neq j$, and $(\lambda_i, \ell_j), 1 \leq i, j \leq p$ (loc. cit. Proposition 2.6 (ii) and Remark 3.2).

We show in this note that the suggested TP_2 property does not hold in general (see section 4), but that the following partial result of this type does hold: if the dimension of the observations equals 2 and $\tilde{\lambda} = (\lambda, \lambda)$, then the density ϕ of $\tilde{\ell}$ can be written $\phi(\tilde{\ell}) = G(\lambda, \tilde{\ell})\phi_0(\tilde{\ell})$, where G satisfies the FKG condition on $\mathbb{R}_+^3$ (we use the notation $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$). This means

$$G(\lambda_1, \tilde{\ell})G(\lambda_2, \tilde{\ell}) \leq G(\lambda_1 \wedge \lambda_2, \tilde{\ell}_1 \wedge \tilde{\ell}_2)G(\lambda_1 \vee \lambda_2, \tilde{\ell}_1 \vee \tilde{\ell}_2), \tag{1.1}$$

for $(\lambda_i, \tilde{\ell}_i) \in \mathbb{R}_+^3, i = 1, 2$. Here we use the conventions $x \wedge y = \min(x, y), x \vee y = \max(x, y)$, if $x, y \in \mathbb{R}$ and $x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n), x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n)$, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. since in our case the function G is strictly positive on $\mathbb{R}_+^3$, proving that G satisfies the FGK condition on $\mathbb{R}_+^3$ is equivalent to proving that G is pairwise TP_2 on $\mathbb{R}_+^3$ (cf. Perlman and Olkin (1980), Remark 2.3). This means that the power function is monotone "on the diagonal" in the 2-dimensional case. We believe that this property holds generally (i.e. also for dimensions higher than 2), but were not able to adapt our method of proof to the higher dimensional case.

The key lemmas in our approach are given in Section 2. They give integral inequalities for diagonal elements of an orthogonal matrix under densities of an exponential type with respect to Haar measure on the orthogonal group. These lemmas are similar in spirit to correlation inequalities for spin configurations in Kelly and Sherman (1968).

The results in Section 3 follow easily from the Lemmas in Section 2 by using the integral representation of the hypergeometric function ${}_0F_1\left(\frac{1}{2}n; \frac{1}{4}\Lambda, L\right)$, where

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2), L = \mathrm{diag}(\ell_1, \ell_2),$$

which is given in James (1961). If $\Lambda = \lambda I$, with $\lambda \geq 0$, this integral reduces to an integral over the orthogonal group $O(n)$ (instead of a *repeated* integral involving the orthogonal groups $O(2)$ and $O(n)$). The density $\phi(\tilde{\ell})$ of the characteristic roots ℓ_1 and ℓ_2 of XX' can then be written

$$\phi(\tilde{\ell}) = G(\lambda, \tilde{\ell})\phi_0(\tilde{\ell}),$$

where

$$G(\lambda, \tilde{\ell}) = {}_0F_1\left(\tfrac{1}{2}n; \tfrac{1}{4}\lambda I, L\right)\exp(-\lambda)$$

and ϕ_0 is the density under the null hypothesis $\mu = 0$. The TP_2 properties of the function G follow from the corresponding properties of the hypergeometric function ${}_0F_1\left(\frac{1}{2}n; \lambda I, L\right)$. The monotonicity result for the power function follows from this by using the FKG inequality due to Fortuin, Ginibre and Kasteleyn (1971). For an exposition on the FKG inequality and its uses we refer to Kemperman (1977) and Perlman and Olkin (1980).

2 Some preparatory lemmas

Lemma 2.1 *Let $a_1 \geq a_2 \geq 0$ and let H be an $n \times n$ orthogonal matrix, where $n \geq 2$. Then the diagonal elements h_{11} and h_{22} have a non-negative covariance under the density*

$$f(h_{11}, h_{22}) = \exp\left\{\sum_{i=1}^{2} a_i h_{ii}\right\} \Big/ \int_{O(n)} \exp\left\{\sum_{i=1}^{2} a_i h_{ii}\right\} dH \tag{2.1}$$

with respect to Haar measure on $O(n)$, where dH denotes Haar measure on $O(n)$.

Proof. First consider the special orthogonal group $SO(n)$ of orthogonal matrices with determinant equal to one. Any $H \in SO(n)$ can be written as a product $H_{n-1} \ldots H_1$ of rotations $H_1, \ldots, H_{n-1}$, where

$$H_k = H^{(1)}(\theta_{1k}) \ldots H^{(k)}(\theta_{kk}) \tag{2.2}$$

and $H^{(i)}(\theta_{ik})$ is a rotation by the angle θ_{ik} in the (x_i, x_{i+1})-plane, oriented such that the rotation from the i-th unit vector e_i to the $(i+1)^{th}$ unit vector e_{i+1} is positive. The range of the angles θ_{ik} is as follows:

$$\begin{cases} 0 \leq \theta_{ik} < 2\pi, & i = 1, \\ 0 \leq \theta_{ik} < \pi, & i > 1. \end{cases} \tag{2.3}$$

These parameters are called *Euler angles*, see e.g. Vilenkin (1968), chapter IX. In terms of these parameters, Haar measure on $SO(n)$ is given by

$$dH = c_n \prod_{k=1}^{n-1} \prod_{j=1}^{k} \sin^{j-1} \theta_{jk} d\theta_{jk} \tag{2.4}$$

where

$$c_n = \prod_{k=1}^{n} \Gamma(k/2)/(2\pi^{k/2}), \tag{2.5}$$

see Vilenkin (1968), p. 439. By induction it is seen that

$$h_{n1} = \prod_{k=1}^{n-1} \sin \theta_{kk}, h_{1n} = (-1)^{n-1} \prod_{k=1}^{n-1} \sin \theta_{k,n-1}. \tag{2.6}$$

Note that the distribution of (h_{11}, h_{22}) under Haar measure on the orthogonal group is the same as the distribution of $(\epsilon_1 h_{n1}, \epsilon_2 h_{1n})$, where ϵ_1 and ϵ_2 are independent random variables with the same distribution $P\{\epsilon_i = 1\} = P\{\epsilon_i = -1\} = \frac{1}{2}$ and (h_{n1}, h_{1n}) is distributed according to Haar measure on $SO(n)$, independent of (ϵ_1, ϵ_2). Thus, taking the expectation with respect to (ϵ_1, ϵ_2), we get

$$
\begin{aligned}
&\int_{O(n)} h_{11} h_{22} f(h_{11}, h_{22})\, dH \\
&= c_1 E \Bigg\{ \epsilon_1 \epsilon_2 \int_0^{2\pi} d\theta_{11} \int_0^{2\pi} d\theta_{1,n-1} \int_0^{\pi} d\theta_{22} \int_0^{\pi} d\theta_{2,n-1} \\
&\qquad \cdots \int_0^{\pi} \prod_{k=1}^{n=1} (\sin\theta_{kk} \sin\theta_{k,n-1}) (\sin\theta_{n-1,n-1})^{n-2} \\
&\qquad\quad \cdot \prod_{k=1}^{n-2} \left(\sin^{k-1}\theta_{kk} \sin^{k-1}\theta_{k,n-1} \right) \\
&\qquad\quad \cdot f\left(\epsilon_1 a_1 \prod_{k=1}^{n-1} \sin\theta_{kk}, \epsilon_2 a_2 \prod_{k=1}^{n-1} \sin\theta_{k,n-1} \right) d\theta_{n-1,n-1} \Bigg\} \\
&= c_2 \int_0^{\pi/2} d\theta_{11} \int_0^{\pi/2} d\theta_{1,n-1} \int_0^{\pi/2} d\theta_{22} \int_0^{\pi/2} d\theta_{2,n-1} \\
&\qquad \cdots \int_0^{\pi/2} \prod_{k=1}^{n-1} (\sin\theta_{kk} \sin\theta_{k,n-1}) \sinh\left(a_1 \prod_{k=1}^{n-1} \sin\theta_{kk} \right) \\
&\qquad\quad \cdot \sinh\left(a_2 \prod_{k=1}^{n-1} \sin\theta_{k,n-1} \prod_{k=1}^{n-2} \sin^{k-1}\theta_{kk} \sin^{k-1}\theta_{k,n-1} \right) \\
&\qquad\quad \cdot \sin^{n-2}\theta_{n-1,n-1}\, d\theta_{n-1,n-1}.
\end{aligned}
$$

Note that for $n = 2$ there is only one parameter θ_{11}, for $n = 3$ there are three parameters $\theta_{11}, \theta_{22}, \theta_{33}, \theta_{13}, \theta_{23}$, etc. The constants c_1 and c_2 are defined by

$$
\begin{aligned}
c_1 = &\Bigg\{ \int_0^{2\pi} d\theta_{11} \int_0^{2\pi} d\theta_{1,n-1} \int_0^{\pi} d\theta_{22} \int_0^{\pi} d\theta_{2,n-1} \\
&\quad \cdots \int_0^{\pi} \prod_{k=1}^{n-2} \left(\sin^{k-1}\theta_{kk} \sin^{k-1}\theta_{k,n-1} \right) \sin^{n-2}\theta_{n-1,n-1}\, d\theta_{n-1,n-1} \Bigg\}^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
c_2 = &\Bigg\{ \int_0^{\pi/2} d\theta_{11} \int_0^{\pi/2} d\theta_{1,n-1} \\
&\quad \cdots \int_0^{\pi/2} \prod_{0}^{\pi/2} \cosh\left(a_1 \prod_{k=1}^{n-1} \sin\theta_{kk} \right) \cosh\left(a_2 \prod_{k=1}^{n-1} \sin\theta_{k,n-1} \right) \\
&\quad\quad \cdot \left(\prod_{k=1}^{n-2} \sin^{k-1}\theta_{kk} \sin^{k-1}\theta_{k,n-1} \right) \sin^{n-2}\theta_{n-1,n-1}\, d\theta_{n-1,n-1} \Bigg\}^{-1}
\end{aligned}
$$

Now let $S = [0, \pi/2]^{2n-3}$ and define the density q on S by

$$q(\theta_{11}, \ldots, \theta_{n-1,n-1}, \theta_{1,n}, \ldots, \theta_{n-2,n-1})$$

$$= c_2 \cosh\left(a_1 \prod_{k=1}^{n-1} \sin\theta_{kk}\right) \cosh\left(a_2 \prod_{k=1}^{n-1} \sin\theta_{k,n-1}\right) \cdot \left\{\prod_{k=1}^{n-2} \sin^{k-1}\theta_{kk} \sin^{k-1}\theta_{k,n-1}\right\} \sin^{n-2}\theta_{n-1,n-1}. \tag{2.7}$$

Let $\tilde{\theta} = (\theta_{11}, \ldots, \theta_{n-1,n-1}, \theta_{1,n-1}, \ldots, \theta_{n-2,n-1})$, and

$$g_1(\tilde{\theta}) = \left(\prod_{k=1}^{n-1} \sin\theta_{kk}\right) \tanh\left(a_1 \prod_{k=1}^{n-1} \sin\theta_{kk}\right), \tag{2.8}$$

$$g_2(\tilde{\theta}) = \left(\prod_{k=1}^{n-1} \sin\theta_{k,n-1}\right) \tanh\left(a_2 \prod_{k=1}^{n-1} \sin\theta_{k,n-1}\right). \tag{2.9}$$

Then

$$\begin{aligned} &\int_{O(n)} h_{11}h_{22}f(h_{11}, h_{22})dH \\ &= \int_0^{\pi/2} d\theta_{11} \ldots \int_0^{\pi/2} \left(\prod_{k=1}^{n-1} \sin\theta_{kk} \sin\theta_{k,n-1}\right) \\ &\qquad \cdot \tanh\left(a_1 \prod_{k=1}^{n-1} \sin\theta_{kk}\right) \tanh\left(a_2 \prod_{k=1}^{n-1} \sin\theta_{k,n-1}\right) q(\tilde{\theta})\, d\theta_{n-1,n-1} \\ &= E\left\{g_1(\theta)g_2(\theta)\right\} \end{aligned} \tag{2.10}$$

where the expectation is taken with respect to the density q on S.

The density q is pairwise TP_2, since $= \frac{\partial^2}{\partial\theta_{ij}\partial\theta_{kl}} \log q(\tilde{\theta}) \geq 0$ for any pair of different components θ_{ij} and θ_{kl} of $\tilde{\theta}$, and since $q > 0$ on S. Thus, again by the fact that $q > 0$ on S, it follows that q satisfies the FKG condition on S (cf. Perlman and Olkin (1980), Remark 2.3). Since g_1 and g_2 are both nondecreasing in each argument on S, the FKG inequality implies

$$E\{g_1(\tilde{\theta})g_2(\tilde{\theta})\} \geq Eg_1(\tilde{\theta})Eg_2(\tilde{\theta}) \tag{2.11}$$

(see e.g. Perlman and Olkin (1980), Remark 2.5). By computations similar to those used in computing $\int_{O(n)} h_{11}h_{22}f(h_{11}, h_{22})\, dH$ it is seen that

$$\int_{O(n)} h_{11}f(h_{11}, h_{22})\, dH = Eg_1(\tilde{\theta}) \tag{2.12}$$

$$\int_{O(n)} h_{22}f(h_{11}, h_{22})\, dH = Eg_2(\tilde{\theta}) \tag{2.13}$$

The result now follows from (2.10) to (2.13). □

Lemma 2.2 *Under the same conditions as in Lemma 2.1, the diagonal elements h_{11} and h_{22} of H satisfy*

$$\int_{O(n)} h_{ii} f(h_{11}, h_{22})\, dH \geq 0, \quad i = 1, 2, \tag{2.14}$$

where f is given by (2.1).

Proof. Using the notation of the proof of Lemma 2.1 we have

$$\begin{aligned}\int_{O(n)} h_{11} f(h_{11}, h_{22})\, dH &= E g_1(\tilde{\theta}) \\ &= \int_S \left(\prod_{k=1}^{n-1} \sin\theta_{kk} \right) \tanh\left(a_1 \prod_{k=1}^{n-1} \sin\theta_{kk} \right) q(\tilde{\theta})\, d\tilde{\theta}, \end{aligned} \tag{2.15}$$

where $S = [0, \pi/2]^{2n-3}$; see (2.7), (2.8) and (2.12). The expression on the right-hand side of (2.15) is clearly non-negative (and strictly positive if $a_1 > 0$). The proof for h_{22} is completely similar. □

3 Total positivity and monotonicity

Theorem 3.1 *Let $L = diag(\ell_1, \ell_2)$ and $\Lambda = diag(\lambda, \lambda)$, where $\ell_i \geq 0, i = 1, 2,,$ and $\lambda > 0$. Then the hypergeometric function ${}_0F_1(\frac{1}{2}n; \frac{1}{4}\Lambda, L)$ is TP_2 in (ℓ_1, ℓ_2) and in $(\ell_j, \lambda), j = 1, 2,$ for each $n \geq 2$.*

Proof. We use the following integral representation

$${}_0F_1\left(\tfrac{1}{2}n; \tfrac{1}{4}\Lambda, L\right) = \int_{O(2)} \int_{O(n)} \exp\Big\{ \mathrm{tr} D'_\lambda H_1 D_\ell H'_2 \Big\} dH_1 dH_2, \tag{3.1}$$

where $H_1 \in O(2), H_2 \in O(n)$ and dH_1 and dH_2 denote Haar measure on $O(2)$ and $O(n)$, respectively; D_ℓ is a $2 \times n$ matrix defined by $(D_\ell)_{ij} = \ell_i^{1/2} \delta_{ij}$ and D_λ is a $2 \times n$ matrix defined by $(D_\lambda)_{ij} = \lambda_i^{1/2} \delta_{ij}$ where δ_{ij} is Kronecker's delta (see e.g. James (1961)). When $\Lambda = \mathrm{diag}(\lambda, \lambda)$ we obtain the following integral representation

$${}_0F_1\left(\tfrac{1}{2}n; \tfrac{1}{4}\Lambda, L\right) = \int_{O(n)} \exp\Big\{ \lambda^{1/2} \sum_{j=1}^{2} \ell_j^{1/2} h_{jj} \Big\} dH \tag{3.2}$$

since in this case

$$\begin{aligned}&\int_{O(n)} \exp\left\{ \mathrm{tr} D'_\lambda H_1 D_\ell H'_2 \right\} dH_2 \\ &= \int_{O(n)} \exp\Big\{ \lambda^{1/2} \sum_{i=1}^{n} \sum_{j=1}^{2} h_{ij}^{(1)} h_{ij}^{(2)} \ell_j^{1/2} \Big\} dH_2 \\ &= \int_{O(n)} \exp\Big\{ \lambda^{1/2} \sum_{j=1}^{2} \ell_j^{1/2} h_{jj} \Big\} dH \end{aligned} \tag{3.3}$$

where $H_1 = (h_{ij}^{(1)})$ and $H_2 = (h_{ij}^{(2)})$. The last equality in (3.3) holds, since

$$\sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij}^{(1)} h_{ij}^{(2)} \ell_j^{1/2} = \operatorname{tr}\left\{\overline{H}_1 A(L) H_2'\right\}, \tag{3.4}$$

where $A(L)$ is the $n \times n$ matrix defined by

$$A(L)_{ii} = \ell_i^{1/2},\ i = 1, 2,$$

and $A(L)_{ij} = 0$ for other values of (i, j), and where $\overline{H}_1$ is the $n \times n$ orthogonal matrix defined by

$$(\overline{H}_1)_{ij} = h_{ij}^{(1)}, 1 \le i, j \le 2, (H_1)_{ii} = 1, i > 2.$$

Here we use that the function

$$\Psi : A \mapsto \int_{O(n)} \exp\{\mathrm{tr} AH\}\, dH,\ A \text{ an } n \times n \text{ matrix},$$

is invariant under transformations $A \mapsto H_1 A, H_1 \in O(n)$.

Let $F = {}_0F_1\left(\frac{1}{2}n; \frac{1}{4}\Lambda, L\right)$. Then

$$\begin{aligned}
&\frac{\partial^2}{\partial \ell_1 \partial \ell_2} \log F \\
&= \tfrac{1}{4}\lambda(\ell_1\ell_2)^{-\frac{1}{2}} \int_{O(n)} h_{11}h_{22} \exp\left\{\lambda^{\frac{1}{2}} \sum_{j=1}^{2} \ell_j^{1/2} h_{jj}\right\} dH/F \\
&-\tfrac{1}{2}\lambda(\ell_1\ell_2)^{-\frac{1}{2}} \left[\int_{O(n)} h_{11} \exp\left\{\lambda^{1/2} \sum_{j=1}^{2} \ell_j^{1/2} h_{jj}\right\} dH/F\right] \\
&\qquad \cdot \left[\int_{O(n)} h_{22} \exp\left\{\lambda^{1/2} \sum_{j=1}^{2} \ell_j^{1/2} h_{jj}\right\} dH/F\right]
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial \lambda \partial \ell_i} \log\ F \\
&= \tfrac{1}{4}(\lambda \ell_i)^{-\frac{1}{2}} \int_{O(n)} h_{ii} \exp\left\{\lambda^{\frac{1}{2}} \sum_{j=1}^{2} \ell_j^{\frac{1}{2}} h_{jj}\right\} dH/F \\
&+\tfrac{1}{4}\ell_i^{-\frac{1}{2}} \int_{O(n)} h_{ii} \sum_{j=1}^{2} \ell_j^{\frac{1}{2}} h_{jj} \exp\left\{\lambda^{\frac{1}{2}} \sum_{j=1}^{2} \ell_j^{\frac{1}{2}} h_{ij}\right\} dH/F \\
&-\tfrac{1}{4}\ell_i^{-\frac{1}{2}} \left[\int_{O(n)} \sum_{j=1}^{2} \ell_j^{\frac{1}{2}} h_{jj} \exp\left\{\lambda_j^{\frac{1}{2}} \sum_{j=1}^{2} \ell_j^{\frac{1}{2}} h_{jj}\right\} dH/F\right] \\
&\qquad \cdot \int_{O(n)} h_{ii} \exp\left\{\lambda^{\frac{1}{2}} \sum_{j=1}^{2} \ell_j^{\frac{1}{2}} h_{jj}\right\} dH/F
\end{aligned} \tag{3.6}$$

By Lemmas 2.1 and 2.2 it follows that 3.5 and 3.6 are nonnegative. Hence F is pairwise TP_2 in (ℓ_1, ℓ_2) and $(\ell_j, \lambda), j = 1, 2$. □

The following corollary shows that the power function is monotone "on the diagonal".

Corollary 3.1 *Let $\tilde{\ell} = (\ell_1, \ell_2)$ be distributed according to the density*

$$\phi_\lambda(\tilde{\ell}) = \exp(-\lambda)\ {}_0F_1\left(\tfrac{1}{2}n; \tfrac{1}{4}\Lambda, L\right) \phi_0(\tilde{\ell}), \tag{3.7}$$

where $\Lambda = \mathrm{diag}(\lambda, \lambda), L = \mathrm{diag}(\ell_1, \ell_2)$,

$$\phi_0(\tilde{\ell}) = \begin{cases} k(\ell_1 - \ell_2)(\ell_1\ell_2)^{\frac{1}{2}(n-3)} \exp\left\{-\tfrac{1}{2}(\ell_1 + \ell_2)\right\}, \ell_1 \geq \ell_2 \geq 0 \\ 0, \text{otherwise} \end{cases} \tag{3.8}$$

and $k > 0$ is a constant such that ϕ_0 is a probability density. Then the function

$$\lambda \to \int_{\mathbb{R}^2} g(\tilde{\ell})\phi_\lambda(\tilde{\ell})d\tilde{\ell}, \lambda \geq 0,$$

is nondecreasing for each g which is nondecreasing in the components ℓ_1 and ℓ_2 of $\tilde{\ell}$.

Proof. Define

$$G(\lambda, \ell_1, \ell_2) = \exp(-\lambda)\ {}_0F_1\left(\tfrac{1}{2}n; \lambda I, L\right). \tag{3.9}$$

Then $G > 0$ on the rectangle $\mathbb{R}^3_+$. Since $\frac{\partial^2}{\partial\ell_1\partial\ell_2} \log G(\lambda, \ell_1, \ell_2) \geq 0$ and $\frac{\partial^2}{\partial\ell_j\partial\lambda} \log G(\lambda, \ell_1, \ell_2) \geq 0$ for each $(\lambda, \ell_1, \ell_2) \in \mathbb{R}^3_+$, it follows that G is pairwise TP_2 on $\mathbb{R}^3_+$. Since $G > 0$ on $\mathbb{R}^3_+$, this implies that G satisfies the FKG condition on $\mathbb{R}^3_+$ (cf. Perlman and Olkin (1980), Remark 2.3). The result now follows from Proposition 2.6 (ii) and Remark 2.7 in Perlman and Olkin (1980). □

4 A Counterexample

We show that the approach to proving monotonicity of the power function by showing that ${}_0F_1(\frac{1}{2}n; \frac{1}{4}, L)$ is pairwise TP_2 (which worked "on the diagonal" in Section 3), fails in general. Take $n = 2, \Lambda = \mathrm{diag}(\lambda, 0), \lambda > 0, L = (\ell_1, \ell_2), \ell_i \geq 0, i = 1, 2$. Then by the same line of argument as used in Lemma 2.1 we have

$$\frac{\partial^2}{\partial\ell_1\partial\ell_2}{}_0F_1\left(\tfrac{1}{2}n; \tfrac{1}{4}, L\right) = \frac{\partial^2}{\partial\ell_1\ell_2}\int_{O(2)}\int_{O(2)} \exp\left\{\mathrm{tr}^{\frac{1}{2}} H_1 L^{\frac{1}{2}} H_2'\right\} dH_1 dH_2$$

$$= \tfrac{1}{4}\lambda(\ell_1\ell_2)^{-\frac{1}{2}} \int_{O(2)}\int_{O(2)} h_{11}^{(1)} h_{11}^{(2)} h_{12}^{(1)} h_{12}^{(2)} \exp\left\{\lambda^{\frac{1}{2}} \sum_{j=1}^{2} h_{1j}^{(1)} h_{1j}^{(2)} \ell_j^{\frac{1}{2}}\right\} dH_1 dH_2$$

$$= \frac{1}{\pi^2}\lambda(\ell_1\ell_2)^{-\frac{1}{2}} \int_0^{\pi/2} d\theta_1 \int_0^{\pi/2} \cos\theta_1 \cos\theta_2 \sin\theta_1 \sin\theta_2$$

$$\cdot \sinh\left(\lambda^{\frac{1}{2}}\ell_1^{\frac{1}{2}} \cos\theta_1 \cos\theta_2\right) \sinh\left(\lambda^{\frac{1}{2}}\ell_2^{\frac{1}{2}} \sin\theta_1 \sin\theta_2\right) d\theta_2,$$

where $H_1 = (h_{ij}^{(1)})$ and $H_2 = (h_{ij}^{(2)})$. Define the density q on $[0, \pi/2]^2$ by

$$q(\theta_1, \theta_2) = k.\cosh\Big(\lambda^{\frac{1}{2}} \ell_1^{\frac{1}{2}} \cos\theta_1 \cos\theta_2\Big) \cosh\Big(\lambda^{\frac{1}{2}} \ell_2^{\frac{1}{2}} \sin\theta_1 \sin\theta_1\Big), \tag{4.1}$$

where $k > 0$ is chosen such that q is a probability and define

$$\begin{aligned} g_1(\theta_1, \theta_2) &= -\cos\theta_1 \cos\theta_2 \tanh\Big(\lambda^{\frac{1}{2}} \ell_1^{\frac{1}{2}} \cos\theta_1 \cos\theta_2\Big) \\ g_2(\theta_1, \theta_2) &= \sin\theta_1 \sin\theta_2 \tanh\Big(\lambda^{\frac{1}{2}} \ell_2^{\frac{1}{2}} \sin\theta_1 \sin\theta_2\Big). \end{aligned} \tag{4.2}$$

The density q clearly satisfies the FKG condition on S and hence, since g_1 and g_2 are both increasing in θ_1 and θ_2 on S, we have by the FKG inequality

$$Eg_1(\theta_1, \theta_2) g_2(\theta_1, \theta_2) \geq Eg_1(\theta_1, \theta_2) Eg_2(\theta_1, \theta_2), \tag{4.3}$$

where the expectation is taken with respect to the density q on S. Moreover, the inequality in 4.3 is *strict* (cf. Perlman and Olkin (1980), Proposition 2.4 (ii)). Let $F = {}_0F_1(1, \Lambda, L)$. Then

$$\begin{aligned} \frac{\partial^2}{\partial\ell_1 \partial_{\ell_2}} \log F &= \left(\frac{\partial^2}{\partial\ell_1 \partial\ell_2} F\right) / F - \frac{\partial F}{\partial\ell_1} \frac{\partial F}{\partial\ell_2} / F^2 \\ &= \tfrac{1}{4} \lambda (\ell_1 \ell_2)^{-\frac{1}{2}} \left(-Eg_1 g_2 + Eg_1 Eg_2\right) < 0, \end{aligned} \tag{4.4}$$

implying that F is *not* TP_2 in the pair (ℓ_1, ℓ_2).

However, it is shown by a completely different method in Perlman and Olkin (1980) that any test of the type described in Section 1 has a power function which is increasing in λ, if $\Lambda = \text{diag}(\lambda, 0)$.

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